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Existence of Solutions for Neutral Functional Integrodifferential Evolution Equations with Non Local Conditions

S. Manimaran[1], T. Gunasekar[2*], G. V. Subramaniyan[3] and F. Paul Samuel[4]

[1]Ramakrishna Mission Vivekananda College, Chennai - 600 004, Tamil Nadu, India; ponsumanimaran@gmail.com
[2]School of Advanced Sciences, Fluid Dynamics Division, VIT University, Vellore - 632 014, Tamil Nadu, India; gunasekar.t@vit.ac.in
[3]Department of Mathematics, Sudharshan College of Engineering, Sathiyamangalam, Pudukkottai - 622 001, Tamil Nadu, India; tguna84@gmail.com
[4]Department of Mathematics and Physics, University of Eastern Africa, Baraton, Eldoret 2500 - 30100, Kenya; paulsamuelphd@yahoo.com

Abstract

We study the existence of mild and strong solutions for nonlinear neutral functional integrodifferential evolution equations with nonlocal conditions in Banach spaces. The results are obtained by using the fractional powers of operators and Sadovskii's fixed point theorem.

Keywords: Mild and Strong Solution, Neutral Equations, Nonlocal Condition, Semigroup

1. Introduction

Neutral differential equations arise in many areas of applied mathematics and for this reason these equations have received much attention in the last few decades. The literature related to ordinary neutral functional differential equations is very extensive. The work in partial neutral functional differential equations with infinite delay was initiated by Hernandez and Henriquez. First-order partial neutral functional differential equations have been studied by different authors. The reader can consult Adimy[1], Hale[13,14] and Wu[25] for systems with finite delay and Hernandez Henriquez[17,18] and Hernandez[16] for the unbounded delay case. Hernandez[15] established the existence results for partial neutral functional differential equations with nonlocal conditions modelled as

$$\frac{d}{dt}(u(t)) + F(t, u_t)) = Au(t) + G(t, u_t), 0 \le t \le T$$

$$u_0 = \phi + q(u_{t_1}, u_{t_2} \dots u_{t_n}) \in \mathbf{\Omega}.$$

Bahuguna and Agarwal[2] studied the approximation of solution to a partial neutral functional differential equation with nonlocal history condition

$$\frac{d}{dt}(u(t)) + g(t, u(t-\tau_1)) + Au(t) = f(t, u(t), u(t-\tau_2)), t > 0,$$

$$h(u) = \phi, \textbf{on}\, [-\tau, 0]$$

in a separable Hilbert space, where $\tau = max\{\tau_1, \tau_2\}$, τ_1, $\tau_2 > 0$. An extensive theory for ordinary neutral functional differential equations which includes qualitative behavior of classes of such equations and applications to biological and engineering processes. Several authors have studied the existence of solutions of neutral functional differential equations in Banach space[2,3,4,6,11,12,13,15,17,18,23]. The nonlocal Cauchy problem for semilinear evolution equations in Banach space was studied first by Byszekswi[7,8,9] where he established the existence and uniqueness of mild and classical solutions. The nonlocal conditions were motivated by physical problems and their importance is discussed in[7–9]. Balachandran et al[3–5,21] studied the nonlocal Cauchy

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*Author for correspondence

problem for various type of nonlinear integrodifferntial equations. In addition, our result can also be regarded as an extension of the corresponding results on classical problem in[10,22].

In this paper, we study the following neutral functional integrodifferential equation with nonlocal condition

$$\frac{d}{dt}[x(t)) + F(t, x(t), x(b_1(t)), \ldots, x(b_n(t)))] + A(t)x(t)$$
$$= G(t, x(t)), x(a_1(t)), \ldots, x(a_m(t)))$$
$$+ \mathrm{K}\left(t, x(t), \int_0^t k(t, s, x(s)ds \right).$$
$$x(0) + g(x) \ = x_0 \tag{1.1}$$

2. Preliminaries

Let –A be the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators $U(t, s)$ defined in the Banach space X. Let $0 \in \rho(A)$, then define the fractionl power A^{α}, for $0 \le \alpha \le 1$, as a closed linear operator on its domain $D(A^{\alpha}(t))$ which is dense in X. Further $D(A(t))$ is a Banach space under the norm

$$\| x \|_{\alpha} = \| A^{\alpha} x \|, \ x \in D(A^{\alpha}(t))$$

which we denote by X_{α}. Then for each $0 \le \alpha \le 1, X_{\alpha} \to X_{\beta}$ for $0 < \beta < \alpha \le 1$ and the imbedding is compact whenever the resolvent operator of A is compact. We assume that

(a) there is a $M \ge 1$ such that $\|U(t, s)\| \le M$, for all $0 \le t \le a$.

(b) for any $a > 0$, there exists a positive constant C_{α} such that

$$\| A_a U(t, s) \| \le \frac{C_a}{t^a}, \ 0 < \mathrm{t} \le \mathrm{T}.$$

Now we represent the basic assumptions on equation (1.1):

(H_1) F:$[0, T] \times X^{(n+1)} \to X$ is a continuous function, $F([0, T] \times X^{(n+1)}) \subset D(A(t))$ with n a positive integer, and there exists consants L, L_1 >0 such that the function $A(t)F$ satisfies the Lipschitz condition:

$$\| A(t)F(s_1, x_0, x_1, \ldots, x_n) - A(t)F(s_2, \overline{x_0}, \overline{x_1}, \ldots, \overline{x_n}) \|$$
$$\le L\left(| s_1 - s_2 | + \max_{i=0,1,\ldots,n} \| x_i - \overline{x_i} \| \right) \tag{2.2}$$

for every $0 \le s_1, s_2 \le T, x_i, \bar{x}_i \in X, i = 0, 1, \ldots, n,$ and the inequality

$$\| A(t)F(t, x_0, x_1, \ldots, x_n) \| \le L_1 \ (\max(\| x_i \| : i = 0, 1, \ldots, n) + 1) \tag{2.3}$$

holds for any $(t, x_0, x_1, ..., x_\mathrm{n}) \in [0, T] \times X^{n+1}$.

(H_2) The function $G : [0, T] \times X^{m+1} \to X$ satisfies the following condition;
  (i) for each $t \in [0, T]$, the function $G(t, .) : X^{m+1} \to X$ is continuous, and for each $(x_0, x_1, ..., x_n) \in X^{m+1}$, the function $G(., x_0, x_1, ..., x_n) : [0, T] \to X$ is strongly measurable;
  (ii) for each positive constant $k \in N$, there is a positive function $g_k \in \mathrm{L}'([0, T])$ such that

$$\sup \| x_0 \|, \ldots, \| x_m \|$$
$$\le k \| G(., x_0, x_1, \ldots, x_m) \| \le g_k(\mathrm{t},)$$

and

$$\lim \inf_{k \to +\infty} \frac{1}{k} \int_0^T g_k(s)\, ds = \gamma < \infty$$

(H_3) The function $K : [0, T\ ] \times X \times X \to X$ satisfies the follwing condition:
  (i) For each $t \in [0, T]$, the function $K(., ., .) : X \times X \to X$ and for each $x, y \in X, K(., x, y) : [0, T] \to X$ is strongly measurable.
  (ii) For each positive number $r \in N$, there is a positive function $\mu_r \in L'([0, T])$ such that

$$\sup_{\|x\| \le r} \left\| K\left(s, x(s), \int_0^s k(\mathrm{s}, \tau, x(\tau)) d\tau \right) \right\| \le \mu_r(\mathrm{s})$$

and

$$\lim in\, f_{r \to +\infty} \frac{1}{r} \int_0^T \mu_r(s) ds = \gamma_1 < \infty$$

(H_4) $a_i, a_j \in C([0, T]; [0, T]), i = 1, 2, ..., m, j = 1, 2, ..., n, g \in C(H; X)$ is completely continuous, where $H = C([0, T]; X)$, and there exists a constant $L_2 > 0$ such that $\|g(x)\| \le L_2 \|x\|$ for each $x \in H$.

THEOREM 2.1 (SADOVSKII'S FIXED POINT THEOREM, CF.[24]).

Let P be a condensing operator on a Banach space X, i.e., P is continuous and takes bounded sets into bounded sets, and $\alpha(P(B)) \le \alpha(B)$ for every bounded set B of x with $\alpha(B) > 0$. If $P(H) \subset H$ for convex, closed and bounded set H of X, then P has a bounded point in H (where $\alpha(.)$ denotes the Kuratowski's measures of non-compactness).

3. Existence of Mild Solutions

DEFINITION 3.1

A continuous function $x(.) : [0, T] \to X$ is said to be a mild solution of the nonlocal Cauchy problem(1.1), is the function $U(t, s)F(s, x(b_1(s)), ..., x(b_n(s)))$, $s \in (0, t)$ in integrale on $[0, t)$ and the following integral equation is verified:

$$\begin{aligned} x(t) = U(t,0)[x_0 + F(0, x(0), x(b_1(0)), \ldots, x(b_n(0))) - g(x)] \\ - \mathrm{F}(t, x(b_1(t)), \ldots, x(b_n(t))) \\ + \int_0^t U(t,s)A(s)F(s, x(s), x(b_1(s)), \ldots, x(b_n(s)))\,ds \\ + \int_0^t U(t,s)G(s, x(s), x(a_1(s)), \ldots, x(a_m(s)))\,ds \\ + \int_0^t U(t,s)\left[K\left(\mathrm{s}, x(s), \int_0^s k(s, \tau, x(\tau))\,ds\right)\right]. \end{aligned}$$

THEOREM 3.1

If the assumption $(H_1) - (H_4)$ are satisfied and $x_0 \in X$, then the nonlocal Cauchy problem (1.1) has a mild solution provided that

$$L_0 : L[(M+1)M_0, MT] < 1$$

$$and\ M_0L_1 + (L_2 + \gamma + \gamma_1 + M_0L_1 + L_1T)M < 1, \qquad (3.4)$$

where M is from property (f), $M_0 = \sup||A^{-1}(t)||$.

PROOF.

For the sake of brevity, we rewrite $(t, x(t), x(b_1(t)), ..., x(b_n(t))) = (t, v(t))$ and $(t, x(t), x(a_1(t)), ..., x(a_m(t))) = (t, u(t))$. Define the operator P on $C([0, T]; X)$ by the formula

$$\begin{aligned} (P_x)(t) = U(t,0)[x_0 + F(0, v(0)) - g(x)] - F(t, v(t)) \\ + \int_0^t U(t,s)A(s)F(s, v(s))\,ds + \int_0^t U(t,s)G(s, u(s))ds \\ + \int_0^t U(t,s)K\left(s, x(s), \int_0^s k(s, \tau, x(\tau))d\tau\right)ds, \quad 0 \le t \le T. \end{aligned}$$

for each positive number k, let $B_k = \{x \in C([0, T]; X) : ||x(t)|| \le k, 0 \le t \le T\}$, then for each k, B_k is clearly a nonempty bounded closed convex set in $X([0, T]; X)$, since the following relation holdes

$$\begin{aligned} ||U(t,s)A(s)F(s, v(s))|| \\ \le ||U(t,s)|| \, ||A(s)F(s, v(s))|| \le ML_1(\mathrm{k}+1), \end{aligned}$$

then from Bouchner's theorem[20] it follows that $U(t, s)A(s) F(s, v(s))$ is integrable on $[0, t]$ since it is obviously strongly measurable, so P is well defined on B_k. We claim that there exists a positive number k such that $P(B_k) \subseteq B_k$. It is not true, then for each positive number k, there is a function $x_k(.) \in B_k$, but $Px_k \notin B_k$, that is $||Px_k(t)|| > k$ for some $t(k) \in [0, T]$. However, on the other hand, we have

$$\begin{aligned} k < ||Px_k(t)|| = \Big\| U(t,0)[x_0 + F(0, v(0)) - g(x_k)] - F(t, v_k(t)) \\ + \int_0^t U(t,s)A(s)F(s, v_k(s))ds \Big\| + \Big\| \int_0^t U(t,s)G(s, u_k(s))ds \\ + \int_0^t U(t,s)K(s, x_k(s)), \int_0^t k(s, \tau, x_k(\tau))d\tau)ds \Big\| \\ \le ||U(t,0)[x_0 - g(x_k) + F(0, v_k(0))]|| \\ + ||A(t)A^{-1}(t)F(t, v_k(t))|| + \int_0^t ||U(t,\mathrm{s})|| \, ||A(s)F(s, v_k(s))||ds \\ + \int_0^t ||U(t,s)|| \, ||G(s, u_k(s))|| \\ + \int_0^t ||U(t,\mathrm{s})|| \Big\| K(s, x_k(s), \int_0^t k(s, \tau, x_k(\tau))d\tau) \Big\| ds \\ \le M[||x_0|| + L_2k + M_0L_1(k+1)] + M_0L_1(k+1) + ML_1(k+1)T \\ + M\int_0^T gk(s)ds + M\int_0^T \mu_r(s)ds \end{aligned}$$

Dividing on both sides by k and taking the lower limit $k \to +\infty$, we get

$$M_0L_1 + (L_2 + M_0L_1 + L_1T + \gamma + \gamma_1)M \ge 1.$$

This is contradicts (7). Hence some positive k, $PB_k \le B_k$.

We will show that the operator P has a fixed point on B_k, which implies that equation (1.1) has a mild solution. To this end, we decompose P into $P = P_1 + P_2$, where the operator P_1, P_2 are defined on B_k respectively by

$$\begin{aligned} (P_1x)(t) = U(t,0)F(0, v(0)) - F(t, v(t)) \\ + \int_0^t U(t,s)A(s)\mathrm{F}(s, v(s))ds \end{aligned}$$

and

$$\begin{aligned} (P_2x)(t) = U(t,0)[x_0 - g(x)] + \int_0^t U(t,s)G(s, u(s))ds \\ + \int_0^t U(t,s)K(s, x(s), \int_0^s k(s, \tau, x(\tau))d\tau)ds, \end{aligned}$$

$0 \le t \le T$, and will verify that P_1 is contraction which P_2 is compact operator.

To prove P_1 is a contraction, we take $x_1, x_2 \in B_k$, then for each $t \in [0, T]$ and by condition (H_1) and (6), we have

$$\begin{aligned}
&\|(P_1x_1)(t)-(P_2x_2)(t)\| \\
&\quad \le \|U(t,0)[F(0,v_1(0))-F(0,v_2(0))]\| \\
&\quad + \|F(t,v_1(t))-F(t,v_2(t))\| \\
&\quad + \left\|\int_0^t U(t,s)A(s)[F(s,v_1(s))-F(s,v_2(s))]ds\right\| \\
&\quad \le (M+1)M_0L \sup_{0\le t\le T} \|x_1(s)-x_2(s)\| \\
&\quad = L_0 \sup_{0\le t\le T} \|x_1(s)-x_2(s)\| \\
&\|(P_1x_1)(t)-(P_2x_2)(t)\| \le L_0\|x_1(s)-x_2(s)\|,
\end{aligned}$$

which shows that P_1 is contraction.

To prove that P_2 is compact, firstly we prove that P_2 is continuous on B_k, Let $\{x_n\} \subseteq B_k$ with $x_n \to x$ is B_k, then by $(H_2)(i)$, we have

$$G(s,u_n(s)) \to G(s,u(s)),\ n\to\infty$$

$$K\left(t, x_n(t), \int_0^t k(t,s,x_n(s))ds\right) \to K\left(t, x(t), \int_0^t k(t,s,x(s))ds\right)$$

as $n \to \infty$. Since $\|G(s, u_n(s)) - G(s, u(s))\| \le 2g_k(s)$,

$$\begin{aligned}
&\left\|K\left(t, x_n(t), \int_0^t k(t,s,x_n(s))ds\right)\right. \\
&\qquad \left.-K\left(t, x(t), \int_0^t k(t,s,x(s))ds\right)\right\| \le 2\mu_r(s),
\end{aligned}$$

then by dominated convergence theorem we have,

$$\begin{aligned}
&\|P_2x_n - P_2x\| = \sup_{0\le t\le T} \|U(t,0)[x_n(0)-x(0)] \\
&\quad + \int_0^t U(t,s)[G(s,u_n(s))-G(s,u(s))]ds\| \\
&\quad + \sup_{0\le t\le T} \left\|\int_0^t U(t,s)[K\left(s, x_n(s), \int_0^s k(s,\tau,x_n(\tau))d\tau\right)ds\right. \\
&\quad \left. -K\left(s, x(s), \int_0^s k(s,\tau,x(\tau))\,d\tau\right)ds]\right\| \to 0 \quad \text{as } n\to\infty.
\end{aligned}$$

That is P_2 is continuous.

We prove that the family $\{P_2x : x \in B_k\}$ is family of equicontinuous functions. To do this, let $0 \le t_1 \le t_2 \le T$, $0 < \in < t_1$, then

$$\begin{aligned}
&\|(P_2x)(t_2)-(P_2x)(t_1)\| \le \|U(t_2,0)-U(t_1,0)\|\|x(0)\| \\
&+\int_0^{t-\in} \|U(t_2,s)-U(t_1,s)\|\|G(\mathrm{s},u(s))\|ds \\
&+\int_{t_1-\in}^{t_1} \|U(t_2,s)-U(t_1,s)\|\|G(\mathrm{s},u(s))\|ds \\
&+\int_{t_1}^{t_2} \|U(t_2,s)\|\|G(\mathrm{s},u(s))\|ds \\
&+\int_0^{t_1-\in} \|U(t_2,s)-U(t_1,s)\|\left\|K\left(\mathrm{s}, x(s)),\ \int_0^s k(s,\tau,x(\tau))d\tau\right)\right\|ds \\
&+\int_{t_1-\in}^{t_1} \|U(t_2,s)\|\left\|K\left(\mathrm{s}, x(s),\ \int_0^s k(s,\tau,x(\tau))d\tau\right)\right\|ds \\
&+\int_{t_1}^{t_2} \|U(t_2,s)\|\left\|K\left(\mathrm{s}, x(s),\ \int_0^s k(s,\tau,x(\tau))d\tau\right)\right\|ds.
\end{aligned}$$

Noting that $\|G(s, u(s))\| \le g_k(s)$ and $g_k(s) \in L'$, we see that $\|(P_2x)(t_2)-(P_2x)(t_1)\|$ tends to zero independently of $x \in B_k$ as $t_2 - t_1 \to 0$ since the compactness of $\{U(t, s), t > s\}$ implies the continuity of $\{U(t, s), t > s\}$ in t in the uniform operator topology uniformly for s. Hence P_2 maps B_k into a family of equicontinuous functions.

It remains to prove that $V(t) = \{(P_2x)(t) : x \in B_y\}$ is relatively compact in X, $V(0)$ is relatively compact in X. Let $0 < t \le T$ be fixed, $0 < \in < t$, for $x \in B_k$, we define

$$\begin{aligned}
&(P_2,\in^x)(t) \\
&\ = U(t,0)\,x(0) + \int_0^{t-\in} U(t,s)\,G(s,u(s))ds \\
&\quad + \int_0^{t_1-\in} U(t,s)\,K\left(s, x(s), \int_0^s k(s,\tau,x(\tau))d\tau\right)ds \\
&\ = U(t,0)\,x(0) + U(t,t-\in)\int_0^{t-\in} U(t-\in,t)\,G(s,u(s))ds \\
&\quad + U(t-\in,s)\int_0^{t_1-\in} U(t-\in,s)K\left(s, x(s), \int_0^s k(s,\tau,x(\tau))d\tau\right)ds
\end{aligned}$$

Then from the compactness of $U(t, s)(t - s > 0)$, we obtain that

$$V_\in(t) = \{(P_2,\in^x)(t) : x \in B_y\}$$

is relatively compact in X for every, $0 < \in < t$. Mortover, $x \in B_y$, we have

$$\begin{aligned}
&\|(P_2x)(t)-(P_2,\in^x)(t)\| \le \int_{t-\in}^t \|U(t,s)\,G(s,u(s))\|ds \\
&+\int_{t-\in}^t \|U(t,s)\|\left\|K(s,x(s), \int_0^s k(s,\tau,x(\tau))\,d\tau)\right\| \\
&\le M\int_{t-\in}^t gk(s)ds + M\int_{t-\in}^t \mu r(s)ds.
\end{aligned}$$

Therefore, there are relatively compact sets arbitrarily close to the set $V(t)$. Hence the set $V(t)$ is also relatively compact in X.

Thus by Arzela-Ascoli theorem P_2 is compact operator. These arguments above enable us to conclude that $P = P_1 + P_2$ is condense mapping on B_k, and by Theorem 2.1 there exists a fixed point $z(.)$ for P on B_k, therefore the nonlocal Cauchy problem (1.1) has mild solution. Then the proof is completed.

4. Existence of Strong Solutions

In this section, we provide conditions which allow the differentiation of the mild solutions obtained in section 3, i.e., these derivatives are shown to satisfy the differential equations of the form(1.1).

DEFINITION 4.2

A function $X(.) : [0, T] \to X$ is said to a strong solution of the nonlocal Cauchy problem(1.1), if

(1) x is continuous on $[0,T]$ and differentiable on $[0,T]$

(2) x satisfies

$$\frac{d}{dt}[x(t)+F(t,x(t),x(b_1(t)),\cdots,x(b_n(t)))]+A(t)x(t)$$
$$=(t,x(t),x(a_1(t)),\cdots,x(a_m(t)))$$
$$+K\left(t,x(t),\int_0^t k(t,s,x(s))ds\right)x(0)+g(x)=x_0$$

THEOREM 4.1

Suppose that condition (H_1), (H_2), (H_3) and (H_4) are satisfied, and additionally the following conditions holds:

(H'_1) For any function $y \in C([0, t]; X)$, the mapping $t \to F(t, x(t), x(b_1(t)), ..., x(b_n(t)))$ is Holders continuous on [0,T];

(H_5) $G(.,.)$ is Holders continuous i.e. there exists a constant $L_3 > o$, $0 < \theta < 1$, such that

$$\|G(s,x_0,\cdots,x_m)-G(\overline{s},\overline{x}_0,\cdots,\overline{x}_m)\|$$
$$\le L_3\left[|s-\overline{s}|^\theta+\max_{i=0,1,\cdots,m}\|x_i-\overline{x}_i\|^\theta\right]$$

For $(s,x_0,\cdots,x_m),(\overline{s},\overline{x}_0,\cdots,\overline{x}_m)\in[0,T]\times X^{m+1}$.

(H_6) There exists a constants L_4, L_5, $L_6 > 0$ such that

$$\|K(t_1,x_1,y_1)-K(t_2,x_2,y_2)\|\le L_4|t_1-t_2|^\theta$$
$$+L_5(\|x_1-x_2\|^\theta+\|y_1-y_2\|^\theta)$$

and $\|k(t, s, x) - k(\tau, s, x)\| \le L_6\|t - \tau\|^\theta$ for any $t, \tau \in [0, T]$ and $x \in X$.

(H_7) There are constants $l_1, l_2 > 0$ such that $|b_j(s)-b_j(\overline{s})| \le l_1|s-\overline{s}|\,|a_i(s)-a_i(\overline{s})|\le l_2|s-\overline{s}|$ for any $s,\overline{s}\in[0,t]$, $i=1,2,\cdots,m$, $j=1,2,\cdots,n$.

(H_8) $x_0 \in D(A)$, $g(x) \in D(A)$ for each $x \in H$, then the nonlocal Cauchy problem (1) has a strong solution on $[0, T]$, provided that (6) and (7).

PROOF.

By Theorem 3.1. We see that equation (1.1) has a mild solution $x(\cdot)$ on $[0, T]$ and we consider the differentiability of $x(t)$.

Let

$$f(t)=F(t,x(t),x(b_1(t)),\cdots,x(b_n(t))),$$
$$\mathrm{o}(t)=U(t,0)[x_0+F(0,,x(0),x(b_1(0)),\cdots,x(b_n(0)))-g(x)]$$
$$=U(t,0)[x(0)+F(0,,x(0),x(b_1(0)),\cdots,x(b_n(0)))]$$
$$p(t)=\int_0^t U(t,s)A(s)F(s,x(s),x(b_1(s)),\cdots,x(b_n(s)))\,ds$$
$$q(t)=\int_0^t U(t,s)G(s,x(s),x(a_1(s)),\cdots,x(a_m(s)))\,ds$$
$$r(t)=\int_0^t U(t,s)\left[k\left(s,x(s),\int_0^s k(s,\tau,x(\tau))ds\right)\right].$$

Then from property (g) and (h) (see section 2) it follows immediately that $o(t)$, $p(t)$, $q(t)$ and $r(t)$ are Holder continuous on $[\in, T]$ for any $0 < \in < T$. Therefore, by condition (H'_1) we obtain that $x(.)$ is Holder continuous on $[\in, T]$. We claim that the Lipschitz continuity of $A(t)F(.,.)$ condition $H(1)$ implies that $A(.)F(.,.)$ is locally Holder continuous. In fact, by assumption (B_2) of $\{A(t) : 0 \le t \le T\}$, we have

$$\|A(s)F(s,v)-A(\overline{s})F(\overline{s},\overline{v})\|\le\|A(s)F(s,v)-A(s)F(\overline{s},\overline{v})\|$$
$$+\|[A(s)-A(\overline{s})A^{-1}(\overline{s})]A(\overline{s})F(\overline{s},\overline{v})$$
$$\le L[|s-\overline{s}|+\max_{i=0,1,\cdots,n}\|x_i-\overline{x}_i\|]$$
$$+E[A(\overline{s})F(\overline{s},\overline{v})]|s-\overline{s}|.$$

This shows that $A(.)F(.,.)$ is locally Holder continuous. Hence condition (H_5), (H_6), (H_7) assure that

$$s\to A(s)F(s,x(s),x(b_1(s)),\cdots,x(b_n(s))),$$
$$s\to G(s,x(s),x(a_1(s)),\cdots,x(a_m(s)))\,ds,$$

and

$$s\to K\left(s,x(s),\int_0^s k(s,\tau,x(\tau))ds\right)$$

are both Holder continuous on $[\in, T]$. Thus, from the proof of Theorem 5.7.1 of19 it is not difficult to see that $p(t) \in D(A)$, $q(t) \in D(A)$, $r(t) \in D(A)$ and

$$p'(t)=A(t)F(t,x(t),x(b_1(t)),\cdots,x(b_n(t)))$$
$$-A(t)\int_0^t U(t,s)A(s)F(s,x(s),x(b_1(s)),\cdots,x(b_n(s)))\,ds$$

$$
\begin{aligned}
q'(t) &= G(t, x(t), x(a_1(t)), \cdots, x(a_m(t))) \\
&\quad - A(t)\int_0^t U(t,s)G(s, x(s), x(a_1(s)), \cdots, x(a_m(s)))\, ds
\end{aligned}
$$

$$
\begin{aligned}
r'(t) &= K\left(t, x(t), \int_0^t k(t,s,x(s))\, ds\right) \\
&\quad - A(t)\int_0^t U(t,s)\left[K\left(s, x(s), \int_0^s k(s,\tau,x(\tau))\, ds\right)\right]
\end{aligned}
$$

so we have that x' satisfies a.e. that

$$
\begin{aligned}
&\frac{d}{dt}[x(t), F(t, x(t), x(b_1(t)), \cdots, x(b_n(t)))] \\
&= \frac{d}{dt}U(t,0)[x_0 + F(0, x(0), x(b_1(0)), \cdots, x(b_n(0))) - g(x)] \\
&\quad + p'(t) + q'(t) + r'(t) \\
&= A(t)\,U(t,0)[F(t, x(t), x(b_1(t)), \cdots, x(b_n(t))) - g(x)] \\
&\quad + A(t)F(t, x(t), x(b_1(t)), \cdots, x(b_n(t))) - A(t)\,p(t) \\
&\quad + G(t, x(t), x(a_1(t)), \cdots, x(a_m(t))) - A(t)q(t) \\
&\quad + K(t, x(t), \int_0^t k(t, s, x(s))\, ds) - A(t)r(t) \\
&= -A(t)x(t) + G(t, x(t), x(a_1(t)), \cdots, x(a_m(t))) \\
&\quad + K(t, x(t), \int_0^t k(t, s, x(s))\, ds).
\end{aligned}
$$

This shows tht $x(\cdot)$ is the strong solution of the nonlocal Cauchy problem (1.1). Thus the proof is completed.

5. References

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