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Dr. Gunasekar T
Vel Tech Rangarajan Dr.Sagunthala R&D Institute of Science and T…


Paul Samuel Francis
University of Eastern Africa, Baraton


M. Mallika Arjunan
SASTRA University

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# EXISTENCE OF SOLUTIONS FOR IMPULSIVE PARTIAL NEUTRAL FUNCTIONAL EVOLUTION INTEGRODIFFERENTIAL INCLUSIONS WITH INFINITE DELAY

T. Gunasekar[1 §], F. Paul Samuel[2], M. Mallika Arjunan[3]

[1]Department of Mathematics
Karunya University
Coimbatore, 641 114, Tamil Nadu, INDIA
[1]School of Advanced Sciences
Fluid Dynamics Division
VIT University
Vellore, 632 014, Tamil Nadu, INDIA
[2]Department of Mathematics and Physics
University of Eastern Africa, Baraton
Eldoret 2500, 30100, KENYA
[3]Department of Mathematics
C.B.M. College
Kovaipudur, Coimbatore, 641 114, Tamil Nadu, INDIA

**Abstract:** This paper investigates a class of impulsive partial neutral functional integrodifferential evolution inclusions with infinite delay in Banach spaces. The existence of mild solutions of these inclusions is determined under the mixed Lipschitz and Caratheodory conditions by using another nonlinear alternative of Leray-Schauder type for multivalued maps due to D. O'Regan. At the end, one example is presented.

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[§]Correspondence author

solution, infinite delay, evolution systems

## 1. Introduction

In this paper we shall consider the problem of existence of solutions of a special class of impulsive partial neutral functional integrodifferential evolution inclusions with infinite delay in Banach spaces described in the form

$$\begin{cases} \frac{d}{dt}[x(t) - g(t, x_t)] \in A(t)x(t) + F(t, x_t, \int_0^t e(t, s, x_s)ds), t \in J = [0, b], \ t \neq t_k, \\ \Delta x(t_k) = I_k(x(t_k^-)), \quad k = 1, 2, ..., m, \qquad 0 < t_1 < t_2 < ... < t_m < b. \\ x_0 = \phi(t) \in \mathcal{B}, \end{cases} \tag{1}$$

where $\{A(t) : t \in J\}$ is the family linear operators defined in Banach space $X$ generating an evolution operator $U(t, s)$. The history $x_t : (-\infty, 0] \to X, x_t(\theta) = x(t+\theta)$, belongs to some abstract phase space $\mathcal{B}$ defined axiomatically; $g : J \times \mathcal{B} \to X$, $F : J \times \mathcal{B} \times X \to \mathcal{P}(X)$, $e : J \times J \times \mathcal{B} \to X$ , $I_k : X \to X, k = 1, 2, ..., m$ are appropriate functions; where $\mathcal{P}(X)$ denotes the class of all nonempty subsets of $X$, $0 = t_0 < t_1 < t_2 < ... < t_m < t_{m+1} = b$, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, $x(t_k^+)$ and $x(t_k^-)$ are respectively the right and left limits of $x$ at $t = t_k$, and $x(t_k^+) = x(t_k)$.

The theory of impulsive partial differential equations has become an important area of investigation in the past two decades because of their applications to various problems arising in communications, control technology, impact mechanics, electrical engineering, medicine, and biology, see the monograph of Laskshmikantham et al. [16] and the papers of Erbe et al. [9], Rogovchenko [22], Liu [18] and the survey papers of Rogovchenko [23], Bainov [1] and the references therein. Recently, the problems of the existence of solutions and controllability of differential equations and differential inclusions have been extensively studied [3, 4, 5, 6, 7, 19] . Benchohra et al. [3, 4] considered the existence of solutions for functional and neutral functional inclusions. Benchohra et al. [5] studied the existence of solutions for integrodifferential inclusions on noncompact intervals. Benchohra et al. [6] discussed the existence of solutions for impulsive multivalued semilinear neutral functional differential inclusions. And Benchohra et al.[7] studied the controllability of semilinear neutral functional differential equations. Liu [19] investigated the controllability of neutral functional differential and integrodifferential inclusions with infinite delay with the aid of a fixed-point theorem for condensing maps to Martelli. However, all these studies are in connection with the ordinary differential systems. Several authors have studied the existence and controllability results for impulsive neutral functional differential and integrodifferential equations with infinite delay [2, 12, 20, 25]. In this paper, we will give the existence of mild solutions for impulsive partial neutral functional integrodifferential inclusions with infinite delay.

Motivated by the above mentioned works, our goal in this paper is to establish the existence results on mild solutions of system (1) by using another nonlinear alternative of Leray-Schauder type for multivalued maps due to D. O'Regan. The rest of this paper is organized as follows. In Section 2, we introduce some notations and necessary preliminaries. In Section 3, we prove the existence results of mild solutions of system (1). The last section is devoted to an example illustrating the abstract theory.

## 2. Preliminaries

In this paper, $X$ will be a seperable Banach space with norm $\|.\|$. $\{A(t) : t \in J\}$ be a family of linear (not necessarily bounded) operators, $A(t) : D(A) \subset X \to X$ $D(A)$ not depending on $t$ and dense subset of $X$ and $U : \Delta = \{(t, s) : 0 \leq s \leq t \leq b\} \to \mathcal{L}(X)$ be the evolution operator generated by the family $\{A(t) : t \in J\}$. There exists some constant $M \geq 1$ such that $\|U(t, s)\| \leq M$, for every $t \in J$. For literature relating to semigroup theory, we suggest [21]. We suppose that $0 \in \rho(A(t))$. For $0 < \alpha \leq 1$, it is possible to define the fraction power $(-A)^{\alpha}(t)$, as a closed linear operator on its domain $D((-A)^{\alpha}(t))$. Furthermore, the subspace $D((-A)^{\alpha}(t))$ is dense in $X$, and the expression $\|x\|_{\alpha} = \|(-A)^{\alpha}(t)\|, x \in D((-A)^{\alpha}(t))$, defines in a norm on $D((-A)^{\alpha}(t))$. Hereafter, let $x_{\alpha}$ denote the Banach space $D((-A)^{\alpha}(t))$ endowed with the norm $\|.\|_{\alpha}$. For $0 < \beta \leq \alpha \leq 1, x_{\alpha} \hookrightarrow x_{\beta}$,and the imbedding is compact whenever the resolvent operator of $A(t)$ is Compact. Also for every $0 < \beta \leq \alpha \leq 1$, there exists a positive constant $C_{\alpha}$ such that

$$\|(-A)^{\alpha}(t)U(t, s)\| \leq \frac{C_{\alpha}}{(t - s)^{\alpha}}, \quad 0 \leq t \leq b$$

We define $\mathcal{PC}$ by the set

$$\Big\{\varphi : [0.b] \to X : \varphi(.) \text{ is continuous at } t \neq t_k,\ \varphi(t_k^+) = \varphi(t_k) \text{ and } \varphi(t_k^-) \text{ exist for } k = 1, 2, ..., m\Big\}.$$

The norm $\|.\|_1$ of the space $\mathcal{PC}$ is defined by $\|\varphi\|_1 = \sup_{0\leq t\leq b}\|\varphi(s)\|$. It is clear that $(\mathcal{PC}, \|\varphi\|_1)$ is a Banach space.

In this paper, we will employ an axiomatic definition of the phase space $\mathcal{B}$ which is similar to that used in [15] ,$\mathcal{B}$ will be linear space of functions mapping $(-\infty, 0]$ to $X$ endowed with a seminorm $\|\varphi\|_{\mathcal{B}}$. We will assume that $\mathcal{B}$ satisfies the following axioms:

(**A**) If $x : (-\infty, b] \to X, b > 0$, such that $x_0 \in \mathcal{B}$ and $x|_{[0,b]} \in \mathcal{PC}$, then for every $t \in [0, b]$ the following conditions hold:

(**i**) $x_t$ is in $\mathcal{B}$,

(**ii**) $\|x(t)\| \leq L\|x_t\|_{\mathcal{B}}$,

(**iii**) $\|x_t\|_{\mathcal{B}} \leq K(t)\sup\{\|x(s)\| : 0 \leq s \leq t\} + M(t)\|x_0\|_{\mathcal{B}}$,

where $L > 0$ is a constant; $K, M : [0,\infty) \to [0,\infty)$, $K$ is continuous, $M$ is locally bounded and $L, K, M$ are independent of $x(\cdot)$.

(**B**) For the function $x(\cdot)$ in $(A), x_t$ is a $\beta$ valued function on $[0, b]$.

(**C**) The space $\mathcal{B}$ is complete.

**Definition 2.1.** The operator $U$ is called the evolution operator generated by the family $\{A(t) : t \in J\}$.

(1) $U(s, s) = I$,

(2) $U(t, r)U(r, s) = U(t, s)$ for all $0 \leq s \leq r \leq t \leq b$.

(3) $(t, s) \to U(t, s)$ is strongly continuous on $\Delta$ and

$$\frac{\partial U(t,s)}{\partial t} = A(t)U(t,s), \quad \frac{\partial U(t,s)}{\partial s} = -U(t,s)A(s).$$

Let $\mathcal{P}(X)$ denote the class of all nonempty subsets of $X$. Let $\mathcal{P}_{bd,cl}(X)$, $\mathcal{P}_{cp,cv}(X)$, $\mathcal{P}_{bd,cl,cv}(X)$ and $\mathcal{P}_{cd}(X)$ denote respectively the family of all nonempty bounded-closed, compact-convex, bounded-closed-convex and compact-acyclic (see [10]) subsets of $X$. For $x \in X$ and $Y, Z \in \mathcal{P}_{bd,cl}(X)$, we define by $D(x, Y) = \inf\{\|x - y\| : y \in Y\}, \rho(Y, Z) = \sup_{a\in Y} D(a, Z)$ and the Hausdorff metric $H : \mathcal{P}_{bd,cl}(X) \times \mathcal{P}_{bd,cl}(X) \to R^+$ by $H(A, B) = \max\{\rho(A, B), \rho(B, A)\}$.

$F$ is called upper semicontinuous (shortly u.s.c) on $X$, if for each $x_* \in X$, the set $F(x_*)$ is nonempty, closed subset of $X$, and if for each open set of $V$ of $X$ containing $F(x_*)$, there exists an open neighborhood $N$ of $x_*$ such that

$F(N) \subseteq X$. $F$ is said to be completely continuous if $F(V)$ is relatively compact, for every bounded subset $V \subseteq X$.

If the multivalued map $F$ is completely continuous with nonempty compact values, then $F$ is u.s.c. if and only if $F$ has a closed graph, (*i.e.* $x_n \to x_*, y_n \to y_*, y_n \in F(x_n)$) imply $y_* \in F(x_*)$.

A point $x_0 \in X$ is called a fixed point of the multivalued map $F$ if $x_0 \in F(x_0)$.

A multivalued map $F : J \to \mathcal{P}_{bd,cl,cv}(X)$ is said to be measurable if for each $x \in X$, the function $t \mapsto D(x, F(t))$ is a measurable function on $J$. For more details on the multivalued maps, see the books of Deimling [8].

**Definition 2.2.** Let $F : X \to \mathcal{P}_{bd,cl}(X)$ be multivalued map. Then $F$ is called a multivalued contraction if there exists a constant $k \in (0, 1)$ such that for each $x, y \in X$ we have

$$H(F(x), F(y)) \leq k\|x - y\|.$$

The constant $k$ is called a contraction constant of $F$.

The consideration of this paper is based on the another nonlinear alternative of Leray-Schauder type for multivalued maps due to D,O'Regan [24].

**Theorem 2.1.** *Let $X$ be a Banach space with $T$ an open, convex subset of $X$ and $u_0 \in T$. Suppose*

$(a)$ *$F : \bar{T} \to \mathcal{P}_{cd}(X)$ has closed graph, and*

$(b)$ *$F : \bar{T} \to \mathcal{P}_{cd}(X)$ is a condensing map with $F(\bar{T})$ a subset of bounded set in $X$ hold. Then either*

- $(i)$ *$F$ has a fixed point in $\bar{T}$; or*
- $(ii)$ *There exists $u \in \partial T$ and $\lambda \in (0, 1)$ with $u \in \lambda F(u) + (1 - \lambda)\{u_0\}$.*

**Definition 2.3.** A multivalued map $F : J \times \mathcal{B} \to \mathcal{P}_{bd,cl,cv}(X)$ is called $L^1$-Caratheodory if

$(i)$ $F(t, x)$ is measurable with respect to $t$ for each $x \in \mathcal{B}$,

$(ii)$ $F(t, x)$ is u.s.c. with respect to $x$ for each $t \in J$, and

$(iii)$ for each $q > 0$, there exists a function $h_q \in L^1(J, [0, \infty))$ such that

$$\|F(t, v) := \sup\{\|g\| : g \in F(t, v)\} \leq h_q(t), a.e.t \in J$$

for all $v \in \mathcal{B}$ with $\|v\|_{\mathcal{B}} \leq q$.

We need the theorem due to Lasota and Opial [17].

**Theorem 2.2.** *Let $X$ be a Banach space, $F$ ba an $L^1$-caratheodory multivalued map with $S_{F,\phi} \neq \phi$ where $S_{F,\phi} := \{g \in L^1(J,X) : g(t) \in F(t,\phi) a.e. t \in J\}$, for each fixed $\phi\mathcal{B}$, and $\mathcal{K}$ be a linear continuous map from $L^1(J,X)$ to $C(J,X)$. Then the operator $\mathcal{K}\ _0S_{F,\phi} : C(J,X) \to \mathcal{P}_{cp,cv}C(J,X)$ is a closed graph operator in $C(J,X) \times C(J,X)$.*

## 3. Main Result

Before starting and proving our main result, we first give the definition of mild solution of system (1).

**Definition 3.4.** A function $x : (-\infty, b] \to X$ is called a mild solution of the system (1) if $x(t) - f(t,x_t)$ is absolutely continuous on $[0,b]/\{t_1,t_2,...,t_m\}, x_0 = \phi(t) \in \mathcal{B}$, the impulsive conditions $x(t_k) = I_k(x(t_k^-)), k = 1,2,...,m$ are satisfied, and for each $s \in [0,t)$, the function $A(s)U(t,s)g(s,x_s)$ is integrable such that

$$\begin{aligned} x(t) = &\ U(t,0)[\phi(0) - g(0,\phi(0))] + g(t,x_t) + \int_0^t A(s)U(t,s)g(s,x_s)ds \\ &+ \int_0^t U(t,s)f(s)ds + \sum_{0<t_k<t} U(t,t_k)I_k(x(t_k^-)), \end{aligned} \tag{3}$$

where $f \in S_{F,x} = \{f \in L^1(J,X) : f(t) \in F(t,x_t, \int_0^t e(t,s,x_s)ds),\ for\ a.e.,\ t \in J\}$.

We consider the following assumptions in the sequel

(H1) $U(t,s)$ is a compact operator whenever $t - s > 0$ and there exists a constant $M > 0$ such that $\|U(t,s)\| \leq M$ for $0 \leq s \leq t \leq b$.

(H2) The multivalued map $F(t,x,y)$ is an $L^1$-caratheodory multivalued map and has compact and convex values for each $(t,x,y) \in J \times \mathcal{B} \times X$.

(H3) There exist constants $0 < \beta < 1$, $L_1, L_2 > 0$ such that $g(.)$ is $x_\beta$-valued, $(-A)^\beta(.)\ g(.)$ is continuous and

- (i) $\|(-A)^\beta(t)\ g(t,x_t)\| \leq L_1\|x_t\|_{\mathcal{B}}, (t,x) \in J \times \mathcal{B}$,
- (ii) $\|(-A)^\beta(t_1)\ g(t_1,x_{1t}) - (-A)^\beta(t_2)\ g(t_2,x_{2t})\| \leq L_2(|t_1 - t_2| + \|x_{1t} - x_{2t}\|_{\mathcal{B}})$,

$(t_i, x_{it}) \in J \times \mathcal{B}, i = 1, 2$, with

$$L_0 := L_2\{\|(-A)^\beta(s)\| + \frac{C_{1-\beta}b^\beta}{\beta}\}K_b < 1.$$

where $K_b = \sup\{K(t) : 0 \le t, s \le b\}$, $M_b = \sup\{M(t) : 0 \le t, s \le b\}$,

(H4) The impulsive function $I_k$ are continuous and there exist positive constants $\beta_k$ such that $\|I_k(x)\| \le \beta_k, k = 1, 2, ..., m$, for each $x \in X$

(H5) There exist a function $q \in L^1(J, R^+)$ such that

$$\| \int_0^t e(t, s, x_s)ds\| \le q(t)\varphi(\|\phi\|_{\mathcal{B}}),$$

for a.e. $t, s \in J$ and all $\phi \in \mathcal{B}$, where $\varphi : R^+ \to (0, \infty)$ is a continuous and nondecreasing function.

(H5) There exist a function $p \in L^1(J, R^+)$ such that

$$\|F(t, \phi, x)\| := \sup\{\|v\| : v(t) \in F(t, \phi, x)\} \le p(t)\varphi(\|\phi\|_{\mathcal{B}}) + \|x\|,$$

for a.e. $t \in J$ and each $\phi \in \mathcal{B}, x \in X$.

**Theorem 3.1.** *Let $\phi \in \mathcal{B}$, if the assumptions (H1)-(H6) are satisfied, then system (1) has atleast one mild solution on $(-\infty, b)$ provided that there exists a constant $N_*$ with*

$$\frac{(1 - K_bL_1\|(-A)^\beta(s)\| - K_bL_1C_{1-\beta}\frac{b^\beta}{\beta})N_*}{N_1 + K_bMb\ \sup_{s\in J} w(s)} > 1, \tag{4}$$

*where* $N_1 = K_bM\|g(0, \phi(0))\| + MK_b\|\phi(0)\| + M_b\|\phi(0)\|_{\mathcal{B}} + MK_b\sum_{k=1}^m \beta_k$.

*Proof.* Let $\mathcal{B}_b$ be the space of all functions $x : (-\infty, b] \to X$ such that $x_0 \in \mathcal{B}$ and the restriction $x|_J \in \mathcal{PC}$. For each $x(t) \in \mathcal{B}_b$. Let $\|.\|_b$ be the semigroup in $\mathcal{B}_b$ defined by

$$\|x\|_b = \|x_0\|_{\mathcal{B}} + \|x\|_1 = \|x_0\|_{\mathcal{B}} + \sup\{\|x(s)\| : 0 \le s \le b\}.$$

Consider the multivalued map $\phi : \mathcal{B}_b \to \mathcal{P}(\mathcal{B}_b)$ defined by $\phi x$, the set of $h \in \mathcal{B}_b$ such that

$$h(t) = \begin{cases} \phi(t), \quad if\ t \in (-\infty, 0] \\ U(t, o)[\phi(0) - g(0, \phi(0))] + g(t, x_t) + \int_0^t A(s)U(t, s)g(s, x_s)ds \\ + \int_0^t U(t, s)f(s)ds + \sum\limits_{0<t_k<t} U(t, t_k)I_k(x(t_k)), \quad t \in J, \end{cases}$$

where $f \in S_{F,x} = \{f \in L^1(J,X) : f(t) \in F(t, x_t, \int_0^t e(t,s,x_s)ds),\ for\ a.e.,\ t \in J\}$. We shall show that the operator $\phi$ has a fixed point, which is then a mild solution of system (1).

$$y(t) = \begin{cases} \phi(t), & if\ t \in (-\infty, 0] \\ U(t,0)\phi(0) & if\ t \in J, \end{cases}$$

Set $x(t) = z(t) + y(t), -\infty < t \leq b$. It is clear that $x$ satisfies (3) if and only if $z$ satisfies $z_0 = 0$ and

$$\begin{aligned} z(t) = & -U(t,0)g(0,\phi(0)) + g(t, z_t + y_t) + \int_0^t A(s)U(t,s)g(s, z_s + y_s)ds \\ & + \int_0^t U(t,s)f(s)ds\ + \sum_{0<t_k<t} U(t,t_k)I_k(z(t_k^-) + y(t_k^-)), \end{aligned}$$

where

$$\begin{aligned} f \in S_{F,x} = \{ & f \in L^1(J,X) : \\ & f(t) \in F(t, z_t + y_t, \int_0^t e(t,s,z_s + y_s)ds),\ for\ a.e.,\ t \in J\}. \end{aligned}$$

Let $\mathcal{B}_b^0$ be the space of all functions $z : (-\infty, b] \to X$ such that $z_0 = 0$ and the restriction $z|_J \in \mathcal{PC}$. For each $z(t) \in \mathcal{B}_b$, let $\|.\|_b$ be the norm in $\mathcal{B}_b^0$ defined by

$$\|z\|_b = \sup\{\|z(s)\| : 0 \leq s \leq b\}.$$

Thus $(\mathcal{B}_b^0, \|.\|_b)$ is a Banach space.

Let the operator $\Phi_1 : \mathcal{B}_b^0 \to \mathcal{P}(\mathcal{B}_b^0)$ be defined by $\Phi_1(x)$, the set of $h_0 \in \mathcal{B}_b^0$ such that

$$h_0(t) = \begin{cases} 0, \quad if\ t \in (-\infty, 0] \\ -U(t,0)g(0,\phi(0)) + g(t, z_t + y_t) + \int_0^t A(s)U(t,s)g(s, z_s + y_s)ds \\ + \int_0^t U(t,s)f(s)ds\ + \sum_{0<t_k<t} U(t,t_k)I_k(z(t_k^-) + y(t_k^-)), \qquad t \in J, \end{cases}$$

We divide the proof into several steps.

*Step 1.* Choose $u_0 = 0$ and a convex and open set $T$ in $\mathcal{B}_b^0$.
Let $\lambda \in (0,1)$ and let $x \in \lambda\Phi_1 x$, then there exists an $f \in S_{F,z}$, such that

$$z(t) = -\lambda U(t,0)g(0,\phi(0)) + \lambda g(t, z_t + y_t) + \lambda \int_0^t A(s)U(t,s)g(s, z_s + y_s)ds$$

$$+\lambda\int_0^t U(t,s)f(s)ds\ +\lambda\sum_{0<t_k<t}U(t,t_k)I_k(z(t_k^-)+y(t_k^-)),\qquad t\in J,$$

By hypotheses (H1)-(H6), we have, for each $t\in J$,

$$\begin{aligned}\|z(t)\|\leq &M\Big(\|g(0,\phi(0))\|+b\ \sup(p(s)+q(s))\chi(\|z_s+y_s\|_{\mathcal{B}})+\sum_{k=1}^m\beta_k\Big)\\ &+L_1\|(-A)^{-\beta}(s)\|\|z_t+y_t\|_{\mathcal{B}}+L_1\int_0^t\frac{C_{1-\beta}}{(t-s)^{1-\beta}}\|z_s+y_s\|_{\mathcal{B}}\ ds.\end{aligned}$$

And from the axiomatic definition of the phase space $\mathcal{B}$, we have

$$\begin{aligned}\|z_t+y_t\|_{\mathcal{B}}\leq&\|z_t\|_{\mathcal{B}}+\|y_t\|_{\mathcal{B}}\\ &\leq K_b\sup\{\|z(s)\|:0\leq s\leq t\}+MK_b\|\phi(0)\|+M_b\|\phi\|_{\mathcal{B}}.\end{aligned}$$

Consider the norm of the function $\chi(t)$, $\|\chi\|=\sup\{\chi(t):0\leq t\leq b\}$. Therefore, we obtain the following inequality

$$\frac{(1-K_bL_1\|(-A)^{\beta}(s)\|-K_bL_1C_{1-\beta}\frac{b^\beta}{\beta})\|\chi\|}{N_1+K_bMb\ \sup_{s\in J}w(s)}\leq 1,$$

Then by (4), there exists $N_*$ such that $\|\chi\|\neq N_*$. Set $T=\{z\in\mathcal{B}_b^0:\|z\|_b<N_*$, from the choice of $T$, there is no $z\in\partial T$ such that $z\in\lambda\Phi_1 z$ for $\lambda\in(0,1)$.

*Step 2.* $\Phi_1$ has a closed graph.

Let $z^n\to z^*, h_0^n\in\Phi_1z^n$ and $h_0^n\to h_0^*$. We shall prove that $h_0^*\in\Phi_1z^*$. Indeed, if $h_0^n\in\Phi_1z^n$ means that there exists $f_n\in S_{F,z^{(n)}}$, such that

$$\begin{aligned}h_0^n(t)=&-U(t,0)g(0,\phi(0))+\int_0^t A(s)U(t,s)g(s,z_s^n+y_s)ds\ +g(t,z_t^n+y_t)\\ &+\int_0^t U(t,s)f(s)ds\ +\sum_{0<t_k<t}U(t,t_k)I_k(z^n(t_k^-)+y(t_k^-)),\qquad t\in J.\end{aligned}$$

We must prove that there exists $f_*\in S_{F,z^*}$, such that

$$\begin{aligned}h_0^*(t)=&-U(t,0)g(0,\phi(0))+\int_0^t A(s)U(t,s)g(s,z_s^*+y_s)ds\ +g(t,z_t^*+y_t)\\ &+\int_0^t U(t,s)f(s)ds\ +\sum_{0<t_k<t}U(t,t_k)I_k(z^*(t_k^-)+y(t_k^-)),\qquad t\in J.\end{aligned}$$

Clearly, we have the norm value of

$$\begin{aligned}&\{h_0^n(t)+U(t,0)g(0,\phi(0)) - g(t,z_t^n+y_t) - \int_0^t A(s)U(t,s)g(s,z_s^n+y_s)ds \\ &\quad - \sum_{0<t_k<t} U(t,t_k)I_k(z^n(t_k^-)+y(t_k^-))\} \\ &\quad - \{h_0^n(t)+U(t,0)g(0,\phi(0)) - g(t,z_t^n+y_t) \\ &\quad - \int_0^t A(s)U(t,s)g(s,z_s^n+y_s)ds - \sum_{0<t_k<t} U(t,t_k)I_k(z^n(t_k^-)+y(t_k^-))\} \\ &\quad \to 0, \text{ as } n\to\infty.\end{aligned}$$

Comsider the linear and continuous operator $\mathcal{K} : L^1(J,X) \to C(J,X)$ defined by

$$f \to \mathcal{K}f(t) = \int_0^t U(t,s)f(s)ds$$

From Theorem 2.2., it follows that $\mathcal{K}\ _0\ S_F$ is a closed graph operator, and

$$\begin{aligned}&h_0^n(t)+U(t,0)g(0,\phi(0)) - \int_0^t A(s)U(t,s)g(s,z_s^n+y_s)ds \ - g(t,z_t^n+y_t) \\ &\quad - \int_0^t U(t,s)f(s)ds \ - \sum_{0<t_k<t} U(t,t_k)I_k(z^n(t_k^-)+y(t_k^-)), \in \mathcal{K}\ _0\ S_{F,z^{(n)}}.\end{aligned}$$

Since $z^{(n)} \to z^*$ and $h_0^n \to h_0^*$, it follows that from Theorem 2.2 that, there exists an $f_* \in S_{F,z^*}$, such that

$$\begin{aligned}h_0^*(t) =& - U(t,0)g(0,\phi(0)) + \int_0^t A(s)U(t,s)g(s,z_s^*+y_s)ds \ + g(t,z_t^*+y_t) \\ &+ \int_0^t U(t,s)f(s)ds \ + \sum_{0<t_k<t} U(t,t_k)I_k(z^*(t_k^-)+y(t_k^-)).\end{aligned}$$

So we can conclude that $\Phi_1$ has closed graph.

We define two the maps. Let the map $\mathcal{A} : T \to \mathcal{B}_b^0$ be defined by

$$h_1(t) = \begin{cases} 0, & \text{if } t\in(-\infty,0], \\ -U(t,0)g(0,\phi(0)) + g(t,z_t+y_t) & \\ + \int_0^t A(s)U(t,s)g(s,z_s+y_s)ds, & \text{if } t \in J, \end{cases}$$

and $\mathcal{C} : T \to \mathcal{P}(\mathcal{B}_b^0)$ be defined by, $\mathcal{C}z$, the set $h_2(t) \in \mathcal{B}_b^0$

$$h_0(t) = \begin{cases} 0, & if\ t \in (-\infty, 0] \\ \int_0^t U(t,s)f(s)ds\ + \sum_{0<t_k<t} U(t,t_k)I_k(z(t_k^-) + y(t_k^-)), & if\ \ t \in J, \end{cases}$$

Then $\Phi_1 = \mathcal{A} + \mathcal{C}$.

*Step 3.* $\mathcal{A}$ is a contraction on $\mathcal{B}_b^0$.
Let $z^1(t), z^2(t) \in T$. By hypothesis (H3), we have

$$\|\mathcal{A}z^1(t) - \mathcal{A}z^2(t)\| \le L_2\{\|(-A)^{-\beta}(t)\| + C_{1-\beta}\frac{b^\beta}{\beta}\}K_b\|z^1 - z^2\|_b$$

Taking supremum over $t$,

$$\|\mathcal{A}z^1(t) - \mathcal{A}z^2(t)\| \le L_o\|z^1 - z^2\|_b, \qquad L_o := L_2\{\|(-A)^{-\beta}(t)\| + C_{1-\beta}\frac{b^\beta}{\beta}\}K_b.$$

This shows that $\mathcal{A}$ is a contraction map, since $L_0 < 1$.

*Step 4.* $\mathcal{C}$ is convex for $z \in T$.
In fact, if $h_2^1(t), h_2^2(t)$ belong to $\mathcal{C}z$, then there exists $f_1, f_2 \in S_{F,x}$ such that

$$h_2^i(t) = \int_0^t U(t,s)f_i(s)ds\ + \sum_{0<t_k<t} U(t,t_k)I_k(z(t_k^-) + y(t_k^-)).$$

Since $F(t,z)$ is convex valued, for $0 \le \tau \le 1, [\tau f_1 + (1-\tau)f_2](t) \in S_{F,x}$. Moreover,

$$(\tau h_2^1(t) + (1-\tau)h_2^2(t)) = \sum_{0<t_k<t} U(t,t_k)I_k(z(t_k^-) + y(t_k^-)) + \int_0^t U(t,s)[\tau f_1(s) + (1-\tau)f_2(s)]ds$$

Therefore, $\mathcal{C}$ is convex for $z \in T$.
It is easy to prove $\mathcal{C}$ maps bounded sets into bounded sets in $\mathcal{B}_b^0$. Also it is obvious that the map $\mathcal{C}$ has closed values.

*Step 5.* $\mathcal{C} : T \to \mathcal{P}(\mathcal{B}_b^0)$ is completely continuous.
First of all, we consider that $\mathcal{C}$ maps bounded sets into equicontinuous sets in $T$. Let $z(t) \in T$. If $h_2(t) \in \mathcal{C}z(t)$, then there exists an $f \in S_{F,z}$, such that, for each $t \in J$

$$h_2(t) = \int_0^t U(t,s)f(s)ds\ + \sum_{0<t_k<t} U(t,t_k)I_k(z(t_k^-) + y(t_k^-)).$$

Let $\tau_1, \tau_2 \in J, \tau_1 < \tau_2$. Then we have

$$\begin{aligned}\|h_2(\tau_2) - h_2(\tau_1)\| \leq & \int_{\tau_1}^{\tau_2} U(\tau_2, s)f(s)ds \; + M \sum_{\tau_1 < t_k < \tau_2} \beta_k \\ & + \{\int_0^{\tau_1 - \epsilon} + \int_{\tau_1-\epsilon}^{\tau_1}\}\|U(\tau_2, s) - U(\tau_1, s)\|\|f(s)\|ds \\ & + \sum_{0 < t_k < \tau_1} \|U(\tau_2, t_k) - U(\tau_1, t_k)\|\beta_k.\end{aligned}$$

We see that $\|h_2(\tau_2) - h_2(\tau_1)\|$ tends to zero as $(\tau_2 - \tau_1) \to 0$ with $\epsilon$ sufficiently small, since $U(t, s)$ is a strongly continuous operator and the compactness of $U(t, s)$ for $t > 0$ implies the continuity in the uniform operator topology. Hence, $\mathcal{C}$ maps bounded sets into equicontinuous sets.

The equicontinuity for the cases $\tau_1 < \tau_2 \leq 0$, or $\tau_1 \leq 0 \leq \tau_2 \leq b$ are similar. Next we shall show that $\mathcal{C}$ maps $T$ into a precompact set. Let $0 < t \leq b$ be fixed and let $\epsilon$ be real number satisfying $0 < \epsilon < t$. For $z \in T$, we define

$$\begin{aligned}h_2^{\epsilon}(t) = U(t, t-\epsilon) & \int_0^{t-\epsilon} U(t-\epsilon, s)f(s)ds \\ & + U(t, t-\epsilon) \sum_{0 < t_k < t-\epsilon} U(t-\epsilon, s)I_k(z(t_k^-) + y(t_k^-)),\end{aligned}$$

where $f \in S_{F,x}$. Since $U(t, s)$ is a compact operator, the set $Y_\epsilon(t) := \{h_2^\epsilon(t) : z \in T\}$ is precompact for every $\epsilon$, $0 < \epsilon < t$. Also, for every $z \in T$, we have

$$\|h_2(t) - h_2^\epsilon(t)\| \leq \int_{t-\epsilon}^{t} U(t, s)f(s)ds \; + M \sum_{t-\epsilon < t_k < t} \beta_k.$$

The right hand side of the above inequality tends to zero as $\epsilon \to 0$. Since there are precompact sets arbitrarily close to the set $Y(t) := \{h_2(t) : z \in T\}$. Hence $Y(t)$ is precompact in $X$. By Arzela-Ascoli theorem, we conclude that $\mathcal{C} : T \to \mathcal{P}(\mathcal{B}_b^0)$ is completely continuous.

As consequence of Steps 3-5, we conclude that $\Phi_1 = \mathcal{A} + \mathcal{C}$ is a condensing map.

All of the conditions of Theorem 2.1 are satisfied. So, system (1) has a mild solution on $(-\infty, b]$. This completes the proof.

## 4. An Example

As an application of our results we consider the impulsive partial neutral integrodifferential inclusion of the form

$$\frac{\partial}{\partial t}\Big[v(t,x) - \int_{-\infty}^{0} k_1(\theta) g_1(t, v(t+\theta, x))d\theta\Big] \in a(t,x)\frac{\partial^2}{\partial x^2}v(t,x)$$
$$+ \int_{-\infty}^{0} p_1(\theta) r_1(t, v(t+\theta,x))d\theta + \int_0^t \int_{-\infty}^{0} p_2(\theta) r_2(t, v(t+\theta, x))dsd\theta, t \in [0,b],$$
$$x \in [0,\pi], \tag{4.1}$$
$$v(t_k^+, x) - v(t_k^-, x) \in [-b_k|v(t_k^-,x)|, +b_k|v(t_k^-,x)|], x \in [0,\pi],$$
$$k = 1,2,...,m \tag{4.2}$$
$$v(t,0) = v(t,\pi), \qquad t \in [0,b], \tag{4.3}$$
$$v(t,x) = \phi(t,x), \qquad x \in [0,\pi],, \ -\infty \le t \le 0, \tag{4.4}$$

where $a(t,x)$ is continuous function and uniformly Holder continuous in $t, b_k > 0,\ k = 1,2,...,m, \phi \in \mathcal{D}$,

$$\mathcal{D} = \{\bar{\psi} : (-\infty, 0] \times [0,\pi] \to R;\ \bar{\psi} \text{ is continuous everywhere except for a countable number of points at which } \bar{\psi}(s^-), \bar{\psi}(s^+) \text{ exists } \bar{\psi}(s^-) = \bar{\psi}(s)\},$$

$0 = t_0 < t_1 < t_2 < ... < t_m < t_{m+1} = b$, $v(t_k^+) = \lim_{(h,x)\to(0^+,x)} v(t_k + h, x)$, $v(t_k^-) = \lim_{(h,x)\to(0^-,x)} v(t_k + h, x)$, $p_1, p_2 : (-\infty, 0] \to R$ a continuous function, $r_1, r_2 : R \times R \to \mathcal{P}_{cv,k}(R)$ a Caratheodory multivalued map.

Let

$$y(t)x = v(t,x), \quad x \in [0,\pi], \quad t \in J = [0,b],$$
$$I_k(y(t_k^-))(x) = [-b_k|v(t_k^-,x)|, +b_k|v(t_k^-,x)|], x \in [0,\pi], \quad k = 1,2,...,m,$$
$$F(t,\phi,\beta_1\phi)(x) = \int_{-\infty}^{0} p_1(\theta) r_1(t, v(t+\theta,x))d\theta + \beta_1\phi(x)$$
$$\beta_1\phi(x) = \int_0^t \int_{-\infty}^{0} p_2(\theta) r_2(t, v(t+\theta,x))dsd\theta$$
$$\phi(\theta)(x) = \phi(\theta, x), \quad -\infty \le t \le 0 \quad x \in [0,\pi].$$

Consider $X = L^2[0,\pi]$ and define $A(t)$ by $A(t)w - a(t,x)w''$ with domain $D(A) = \{w \in X : w, w' are absolutely continuous, w'' \in X, w(0) = w(\pi) = 0\}$.

Then $A(t)$ generates an evolution system $U(t,s)$ satisfying assumption (H5) (see [11]). We can show that problem (4.1)-(4.4) is an abstract formulation of problem (1). Under suitable conditions, the problem (1) has at least one mild solution.

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