

# Existence Solutions for Quasilinear Evolution Integrodifferential Equations with Infinite Delay

Francis Paul Samuel, Tumaini RukikoLisso, Kayiita Zachary Kaunda

**Abstract:** The paper is concerned with the existence and uniqueness solution of quasi linear evolution integrodifferential equations with in finite delay in Banach spaces. The results a reobtained by  $C_0$ -semi group of linear bounded operator and Banach fixed point theorem.

**Keywords:** semigroup; mild and classical solution; Banach fixed point theorem; Infinitedelay.

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## I. INTRODUCTION

Quasilinearevolution equations form savery important class of evolution equations as many time dependent phenomena inphysics,chemistry, biology and engineering can be represented by such evolution equations. Several authorshave studied the existence of solutions of abstract quasilinear evolution equations in Banach space [1,2,3,7,8,9,10,12,14]. Oka [10]and Okaand Tanaka[11]discussed the existence of solutions of quasilinear integrodifferential equations in Banach spaces. Kato [6] studied the non homo geneousevolution equations and Balachandran and Paul Samuel[2]proved the existence and uniqueness of mild and classical delay integro differential equation with non local condition. The problem of existence of solutions of evolution equations in Banach space has been studied by several authors [4,5,8,9].The aim of this paper is to prove the existence and uniqueness of midland classical solutions of quasilinear functional integro differential system with infinitedelay

$$u'(t) + A(t, u)u(t) = f(t, u_t) + \int_0^t k(t-s)g(s, u_s)ds \quad (1.1)$$

$$u_0 = \varphi, \text{ on } [-\mu, 0] \quad (1.2)$$

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where  $u_t(\theta) = u(t + \theta)$ ,  $\theta \in [-\mu, 0]$ , For  $t \in [0, T]$ , we denote by  $E_t$  the Banach space of all continuous functions from  $[-\mu, t]$  to  $X$  endowed with the supremum norm

$$\|\chi\|_{E_t} = \sup_{-\tau \leq \theta \leq t} \|\chi(\theta)\|_X, \chi \in E_t.$$

Let the functions  $f: [0, T] \times E_0$  to  $X$ ;  $g: [0, T] \times E_0$  to  $X$  and  $k: [0, T] \rightarrow [0, T]$  is a real value dcontinuous function. Here we see that  $t \in E_0$ , we assume that for  $u \in E_T$ ,  $f(\cdot, u_{(\cdot)})$  and  $g(\cdot, u_{(\cdot)}) : [0, T] \rightarrow X$  are bounded  $L^1$  function. Further assume that there is a subset  $B$  of  $X$  such that for  $(t, u) \in [0, T] \times E_T$  with  $u(t) \in B$  for  $t \in [0, T]$ ,  $A(t, u)$  is a linear operator in  $X$ . Also  $\varphi \in E_0$  is Lipschitz continuous with constant  $L_\varphi$ . The results obtained in this paper are generalization so the results given by Pazy [13], Kato [6,7] and Balachandran and Paul Samuel [3].

## II. PRELIMINARIES

Let  $X$  and  $Y$  be two Banach spaces such that  $Y$  is densely and continuous ly embedded in  $X$ . For any Banach spaces  $Z$  then norm of  $Z$  is denoted by  $\|\cdot\|$  or  $\|\cdot\|_Z$ . The space of all bounded linear operators from  $X$  to  $Y$  is denoted by  $B(X, Y)$  and  $B(X, X)$  is written as  $B(X)$ . We recall some definitions and known facts from Pazy [13].

**Definition 2.1.** Let  $S$  be a linear operator in  $X$  and let  $Y$  be a subspace of  $X$ . The operator  $S^{\sim}$  defined by  $D(S^{\sim}) = \{x \in D(S) \cap Y : Sx \in Y\}$  and  $S^{\sim}x = Sx$  for  $x \in D(S^{\sim})$  is called the part of  $S$  in  $Y$ .  $\{x + y : x \in B, y \in E\}$ .

**Definition 2.2.** Let  $B$  be a subset of  $X$  and for every  $0 \leq t \leq T$  and  $b \in B$ , let  $A(t, b)$  be the infinitesimal generator of a  $C_0$ -semigroup  $S_{t, b}(s)$ ,  $s \geq 0$ , on  $X$ . The family of operators  $\{A(t, b)\}$ ,  $(t, b) \in [0, T] \times B$ , is stable if there are constants  $M \geq 1$  and  $b$ , known as stability constants, such that  $\rho(A(t, b)) \supset (b, \infty) \cap f$  or  $(t, b) \in [0, T] \times B$ , where  $\rho(A(t, b))$  is the resolvent set of  $A(t, b)$  and

$$\left\| \prod_{j=1}^k R(\lambda : A(t_j, b_j)) \right\| \leq M(\lambda - \omega)^{-k} \text{ for } \lambda > \omega \text{ every finite sequences } 0 \leq t_1 \leq t_2 \leq \dots \leq t_k \leq T, b_j \in B.$$

**Definition 2.3.** Let  $S_{t, b}(s)$ ,  $s \geq 0$  be the  $C_0$ -semigroup generated by  $A(t, b)$ ,  $(t, b) \in I \times B$ . A subspace  $Y$  of  $X$  is called  $A(t, b)$ -admissible if  $Y$  is invariant subspace of  $S_{t, b}(s)$  and the restriction of  $S_{t, b}(s)$  to  $Y$  is a  $C_0$ -semigroup in  $Y$ .

Let  $B \subset X$  be a subset of  $X$  such that for every  $(t, b) \in [0, T] \times B$ ,  $A(t, b)$  is the infinitesimal generator of a  $C_0$ -semigroup  $S_{t, b}(s)$ ,  $s \geq 0$  on  $X$ . We make the following assumptions: (E1) The family  $\{A(t, b)\}$ ,  $(t, b) \in [0, T] \times B$  is stable.

(E2)  $Y$  is  $A(t, b)$  – admissible for  $(t, b) \in [0, T] \times B$  and the family  $\{A(t, b)\}, (t, b) \in [0, T] \times B$  of parts  $A(t, b)$  of  $A(t, b)$  in  $Y$ , is stable in  $Y$ .

(E3) For  $(t, b) \in [0, T] \times B, D(A(t, b)) \supset Y, A(t, b)$  is a bounded linear operator from  $Y$  to  $X$  and  $t \rightarrow A(t, b)$  is continuous in the  $B(Y, X)$  norm  $\|\cdot\|$  for every  $b \in B$ .

(E4) There is a constant  $L > 0$  such that

$$\|A(t, b_1) - A(t, b_2)\|_{Y \rightarrow X} \leq L \|b_1 - b_2\|_X$$

holds for every  $b_1, b_2 \in B$  and  $0 \leq t \leq T$ . Let  $B$  be a subset of  $X$  and  $\{A(t, b)\}, (t, b) \in [0, T] \times B$  be a family of operators satisfying the conditions (E1)-(E4). If  $u \in C([0, T] : X)$  has value in  $B$  then there is a unique evolution system  $U_u(t, s), 0 \leq s \leq t \leq T$ , in  $X$  satisfying, (see [13, Theorem 5.3.1 and Lemma 6.4.2, pp. 135, 201-202])

$$(i) \|U_u(t, s)\| \leq M e^{\omega(t-s)} \text{ for } 0 \leq s \leq t \leq T.$$

where  $M$  and  $\omega$  are stability constants.

$$(ii) \frac{\partial^+}{\partial t} U_u(t, s)w = A(s, u(s))U_u(t, s)w \text{ for}$$

$w \in Y$ , for  $0 \leq s \leq t \leq T$ .

$$(iii) \frac{\partial}{\partial s} U_u(t, s)w = -U_u(t, s)A(s, u(s))w \text{ for}$$

$w \in Y$ , for  $0 \leq s \leq t \leq T$ .

(E5) For every  $u \in C([0, T] : X)$  satisfying  $u(t) \in B$  for  $0 \leq t \leq T$ , we have

$$U_u(t, s)Y \subset Y, \quad 0 \leq s \leq t \leq T \text{ and}$$

$$U_u(t, s) \text{ is strongly continuous in } Y \text{ for } 0 \leq s \leq t \leq T$$

(E6) Every closed convex and bounded subset of  $Y$  is also closed in  $X$ .

Further we assume that

(E7)  $f : [0, T] \times E_0$  to  $X$  is continuous and there exist constants  $F_L > 0$  and  $F_0 > 0$  such that

$$\|f(t, \varphi_1) - f(t, \varphi_2)\| \leq F_L(|t - s| + \|\varphi_1 - \varphi_2\|)$$

$$F_0 = \max \|f(t, u_0)\|.$$

(E8)  $g : [0, T] \times E_0$  to  $X$  is continuous and there exist constants  $G_L > 0$  and  $G_0 > 0$  such that

$$\int_0^t \|g(t, \phi_1 - g(s, \phi_2))\|_X ds \leq G_L (|t - s| + \|\phi_1 - \phi_2\|)_{E_0},$$

$$G_0 = \max \int_0^t \|g(s, u_0)\| ds.$$

(E9) Thereal-valued function  $k$  is continuous on  $[0, T]$  and there exists a positive constant  $K_T$  such that

$$\|k(t)\| \leq K_T \text{ for } t \in [0, T]. \text{ We note that the condition}$$

(E6) is always satisfied if  $X$  and  $Y$  are reflexive Banach spaces.

Next we prove the existence of local classical solutions of the quasilinear problem (1.1)-(1.2).

For a mild solution of (1.1)-(1.2) we mean a function  $u \in E_T$  with values in  $B$  satisfying the integral equation

$$u(t) = U_u(t, 0)\phi(0) + \int_0^t U_u(t, s)[f(s, u_s)] \quad (2.1)$$

$$+ \int_0^s k(s - \tau)g(\tau, u_\tau)] ds$$

$$u_0 = \phi \text{ on } [-\mu, 0].$$

A function  $u \in E_T$  such that  $u(t) \in Y \cap B$  for  $t \in (0, T], u \in C^1((0, T] : X)$  and satisfies (1.1)-(1.2) in  $X$  is called a classical solution of (1.1)-(1.2) on  $[0, T]$ , where  $u \in C^1(0, T] : X$ , space of all continuously differentiable functions from  $[0, T]$  to  $X$  and  $Y$  is a  $A(t, b)$  – admissible subspace of  $X$ .

Further there exists a constant  $E_0 > 0$  such that for every  $u, v \in C([0, T] : X)$  with values in  $B$  and every  $w \in Y$ , we have

$$\|U_u(t, s)w - U_v(t, s)w\| \leq E_0 \|w\|_Y \int_s^t \|u(\tau) - v(\tau)\| d\tau. \quad (2.2)$$

### III. EXISTENCE RESULTS

In this section we prove the existence and uniqueness

result for a classical solution to (1.1)-(1.2). Let  $\tilde{\varphi} \in E_T$  be given by  $\tilde{\varphi}(t) = \varphi(t)$  for  $t \in [-\mu, 0]$  and  $\tilde{\varphi}(t) = \varphi(0)$  for  $t \in [0, T]$ . Denote

$$B_r(\phi(0)) = \{x \in X : \|x - \phi(0)\|_X \leq r\},$$

$$B_{2r}(\tilde{\phi}_0) = \{\chi \in E_0 : \|\chi - \tilde{\phi}_0\|_{E_0} \leq 2r\}.$$

**Theorem 2.1** Let  $B$  and  $V$  be open subsets of  $X$  and  $E_0$  respectively and the family  $A(t, b)$  of linear operators for  $t \in [0, T], b \in B_r(\varphi(0))$  satisfying assumptions (E1)-(E9) and  $A(t, b)\varphi(0) \in Y$  with

$$\|A(t, b)\phi(0)\|_Y \leq C_A, \quad C_A > 0$$

for all  $(t, b) \in [0, T] \times B$ . There exists a positive constant  $T_0$  such that the quasilinear problem (1.1)-(1.2) has a unique classical solution.

$$\|U_u(t, s)\|_{B(Y)} \leq K_1,$$

Take  $r > 0$  such that  $B_r(\varphi(0)) \subset B$  and  $B_{2r}(\tilde{\varphi}_0) \subset V$ .

**Proof:**

Let  $S$  be an empty closed subset of  $C([0, T] : X)$  defined by

$$S = \{\psi \in C_{T_0} : \psi_0 = \varphi, \text{ for } t \in [-\mu, 0], \psi(t) \in B_r(\varphi(0))\}.$$

We easily deduce that  $S$  is a closed, convex and bounded subset of  $C_{T_0}$ . Take  $\psi \in S$ . Now for  $\theta \in [-\mu, 0]$ , we have the following two cases.

Case (i): If  $t + \theta \leq 0$  we have

$$\begin{aligned} \|\psi_t(\theta) - \tilde{\phi}_0(\theta)\|_X &= \|\psi(t+\theta) - \tilde{\phi}(\theta)\|_X \\ &= \|\phi(t+\theta) - \phi(\theta)\|_X \\ &= L_\psi T_0 \leq r. \end{aligned}$$

Case(ii): If  $t + \theta \geq 0$  we have

$$\begin{aligned} \|\psi_t(\theta) - \tilde{\phi}_0(\theta)\|_X &= \|\psi(t+\theta) - \tilde{\phi}(\theta)\|_X \\ &= \|\phi(t+\theta) - \phi(0)\|_X \\ &= \|\phi(0) - \phi(\theta)\|_X \\ &= r + L(-\theta) \\ &\leq r + L_\psi t \\ &\leq r + L_\psi T_0 \leq 2r. \end{aligned}$$

(since  $-\theta \leq t \leq T_0$ ).

Thus, for  $\psi \in S$ ,  $\psi_t \in B_{2r}(\phi)$ . Define  $H: S \rightarrow S$  by

$$Hu(t) = \begin{cases} U_u(t, 0)\phi(0) \\ + \int_0^t U_u(t, s)[f(s, u_s) \\ + \int_0^s k(s-\tau)g(s, u_\tau)d\tau]ds, t \in [0, T_0] \\ \phi(t), & t \in [-\mu, 0]. \end{cases}$$

First we show that  $H$  is well defined and  $Hu(0) = \phi(0)$ . For  $t \geq 0$ , we have

$$\begin{aligned} Hu(t) - \phi(0) &= U_u(t, 0)\phi(0) - \phi(0) \\ &+ \int_0^t U_u(t, s)[f(s, u_s) + \int_0^s k(s-\tau)g(s, u_\tau)d\tau]ds \end{aligned}$$

Taking the norm, we get

$$\begin{aligned} \|Hu(t) - \phi(0)\| &\leq \|U_u(t, 0)\phi(0) - \phi(0)\|_X \\ &+ \int_0^t \|U_u(t, s)[f(s, u_s) + \int_0^s k(s-\tau)g(s, u_\tau)d\tau]\|_X ds \end{aligned}$$

Integrating (ii), we obtain

$$U_u(t, 0)\phi(0) - \phi(0) = \int_0^t U_u(t, s)A(s, u(s))\phi(s)ds.$$

Thus, we have

$$\begin{aligned} \|U_u(t, 0)\phi(0) - \phi(0)\|_X &\leq \int_0^t \|U_u(t, s)A(s, u(s))\|_X \|\phi(s)\|_X ds \\ &\leq C_A K_1 T_0 \\ &\leq \frac{r}{2}. \end{aligned}$$

Also we have

$$\begin{aligned} &\int_0^t \|U_u(t, s)[f(s, u_s) + \int_0^s k(s-\tau)g(s, u_\tau)d\tau]\|_X ds \\ &\leq K_1 \int_0^t [\|f(s, u_s) - f(s, u_0) + f(s, u_0)\| \\ &+ \|\int_0^s k(s-\tau)[g(\tau, u_\tau) - g(\tau, u_0) + g(\tau, u_0)]d\tau\|] ds \\ &\leq K_1 \int_0^t [\|f(s, u_s) - f(s, \phi)\|_X + \|f(s, u_0)\| \\ &+ K_T \int_0^s (\|g(\tau, u_\tau) - g(\tau, \phi)\|)d\tau] ds \\ &\leq K_1 [2r(F_L + K_T G_L) + F_0 + K_T G_0] T_0 \\ &\leq \frac{r}{2}. \end{aligned}$$

Using the result for  $u \in S$ ,  $u_s \in B_{2r}(\phi)$ . Thus, for  $u \in S$  and  $t \geq 0$ , we get

$$\|Hu(t) - \phi(0)\|_X \leq \frac{r}{2} + \frac{r}{2} \leq r.$$

So,  $H$  is well defined for  $u, v \in S$ , we consider

$$\begin{aligned} Hu(t) - Hv(t) &= U_u(t, 0)\phi(0) - U_v(t, 0)\phi(0) \\ &+ \int_0^t \|U_u(t, s)[f(s, u_s) + \int_0^s k(s-\tau)g(s, u_\tau)d\tau]\|_X ds \\ &- \int_0^t \|U_v(t, s)[f(s, v_s) + \int_0^s k(s-\tau)g(s, v_\tau)d\tau]\|_X ds \end{aligned}$$

Let

$$\begin{aligned} I_1 &= \|U_u(t, 0)\phi(0) - U_v(t, 0)\phi(0)\|_X \\ &\leq E_0 \|\phi(0)\| \int_0^t \|u(s) - v(s)\|_X ds \\ &\leq E_0 \|\phi(0)\|_X \|u - v\| T_0. \end{aligned}$$

Also let

$$I_2 = \int_0^t \|U_u(t, s)[f(s, u_s) + \int_0^s k(s-\tau)[g(s, u_\tau)d\tau] - U_u(t, s)[f(s, v_s) + \int_0^s k(s-\tau)[g(s, v_\tau)d\tau] + U_v(t, s)[f(s, v_s) + \int_0^s k(s-\tau)[g(s, u_\tau)d\tau]ds\| \leq K_1 \int_0^t \|f(s, u_s) - f(s, v_s)\| + \int_0^s k(s-\tau)[g(s, u_\tau) - g(s, v_\tau)]d\tau\| ds + E_0 \int_0^t [\|f(s, v_s) + \int_0^s k(s-\tau)g(\tau, u_\tau)d\tau\| \times \int_s^t \|u(\tau) - v(\tau)\|_X d\tau] ds \leq K_1[F_L \int_0^t \|u_s - v_s\| ds + K_T \int_0^t G_L \|u_s - v_s\| ds] + E_0[2r(F_L + K_T G_L) + F_0 K_T G_0] \|u - v\|_{T_0}^2 \leq K_1[F_L \int_0^t \sup_\theta \|u(s+\theta) - v(s+\theta)\|_X ds + K_T G_L \int_0^t \sup_\theta \|u(s+\theta) - v(s+\theta)\|_X ds + E_0[2r(F_L + K_T G_L) + F_0 + K_T G_0] \|u - v\|_{T_0}^2] \leq K_1 F_L T_0 \|u - v\| + K_1 K_T G_L T_0 \|u - v\| + E_0[2r(F_L + K_T G_L) + F_0 + K_T G_0] T_0^2 \|u - v\| \leq (K_1[F_L + K_T G_L] + T_0 E_0[2r(F_L + K_T G_L) + F_0 + K_T G_0]) T_0 \|u - v\|.$$

$$u'(t) + A(t, u)u(t) = f(t, u_t) + \int_0^t k(t-s)g(s, u_s)ds \quad (3.1)$$

$$u(0) = \phi(0). \quad (3.2)$$

Denote  $\tilde{A}(t) = A(t, u(t))$ ,  $\tilde{F}(t) = f(t, u_t)$  and  $\tilde{G}(t) = g(t, u_t)$ , then equation (3.1) can be written as

$$u'(t) + \tilde{A}(t, u)u(t) = \tilde{F}(t) + \int_0^t k(t-s)\tilde{G}(s)ds \quad (3.3)$$

$$u(0) = \phi(0). \quad (3.4)$$

where  $u$  is the unique solution fixed point of  $H$  in  $S$ . Now we assume that (E7)-(E9) we have

$$\|f(t, \chi) - f(s, \chi)\|_X \leq F_L |t - s|,$$

$$\int_0^t \|g(t, \chi) - g(s, \chi)\|_X ds \leq G_L |t - s|$$

and

$$\|k(t)\| \leq K_T.$$

Hence for each  $\epsilon > 0$  there exists a  $\delta > 0$  such that if

$$|t - s| \leq \delta \text{ implies that } \|f(t, \chi) - f(s, \chi)\|_X \leq \epsilon,$$

$$\int_0^t \|g(t, \chi) - g(s, \chi)\|_X ds \leq \epsilon.$$

Thus,  $f(t, \chi) \in E_{T_0}$  and  $g(t, \chi) \in E_{T_0}$  for fixed  $\chi$ . Hence from Pazy [[13] Theorem 5.5.2], we get a unique function  $v \in C^1([0, T_0]; X)$  satisfying (3.3)-(3.4) in  $X$  and  $v$  given by

$$v(t) = U_u(t, 0)\phi(0) + \int_0^t U_u(t, s)[f(s, u_s) + \int_0^s k(s-\tau)g(s, u_\tau)d\tau]ds, \quad t \in [0, T_0],$$

where  $U_u(t, s), 0 \leq s \leq t \leq T_0$  is the evolution system generated by the family  $A(t, u(t)), t \in [0, T_0]$ . The uniqueness of  $v$  implies that  $v \equiv u$  on  $[0, T_0]$ . Thus  $u$  is a unique local classical solution of (1.1)-(1.2).

Hence,

$$I_1 + I_2 = \|Hu(t) - Hv(t)\| \leq (E_0 \|\phi(0)\|_X + K_1[F_L + K_T G_T] + T_0 E_0[2r(F_L + K_T G_T) + F_0 + K_T G_0]) T_0 \|u - v\| \leq \Omega T_0 \|u - v\|_{E_{T_0}} \leq \frac{1}{n} \|u - v\|_{E_{T_0}}.$$

Thus  $H$  is a contraction from  $S$  to  $S$ . So, by the Banach contraction mapping theorem,  $H$  has a unique fixed point  $u \in S$  which satisfies the integral equation. Hence it is a mild solution of (1.1)-(1.2). Now, we consider the following evolution equation

REFERENCES

- [1] H. Amann, Quasilinear evolution equations and parabolic systems, Trans. Amer. math. Soc. 29 (1986), 191-227.
- [2] K. Balachandran and F. Paul Samuel, Existence of solutions for quasilinear delay integrodifferential equations with nonlocal conditions, Electron. J. Diff. Eqns. Vol. 2009(2009), No. 6, 1-7.
- [3] K. Balachandran and K. Uchiyama, Existence of solutions of quasilinear integrodifferential equations with nonlocal condition, Tokyo. J. Math. 23 (2000), 203-210.
- [4] O. Capas, Evolution equations in scales of Banach spaces, Verlag, Teubner Verlag, 2002.
- [5] A. G. Kartsatos, The existence of bounded solutions on the real line of perturbed nonlinear evolution equations in general Banach spaces, Non linear Anal., 17(1991), 1085-1092.
- [6] S. Kato, Nonhomogeneous quasilinear evolution equations in Banach spaces, Nonlinear Anal. 9 (1985), 1061-1071.
- [7] T. Kato, Nonlinear semigroups and evolution equations in Banach spaces, J. Math. Soc. Japan, 19(1967), 508-520.
- [8] T. Kato, Quasilinear equations of evolution with applications to partial differential equations, Lecture Notes in Math. 448(1975), 25-70.

- [9] [9]T. Kato, Abstract evolution equation linear and quasilinear, revisited, Lecture Notes in Math. 1540(1993), 103-125.
- [10] [10]H. Oka, Abstract quasilinear Volterra integrodifferential equations, Nonlinear Anal. 28 (1997), 1019-1045.
- [11] [11]H. Oka and N. Tanaka, Abstract quasilinear integrodifferential equations of hyperbolic type, Nonlinear Anal. 29(1997), 903-925.