Existence of Solutions to Quasilinear Delay Differential Equations with Nonlocal Conditions

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Abstract

We prove the existence and uniqueness of mild and classical solution to a quasilinear delay differential equation with nonlocal condition. The results are obtained by using C_0-semigroup and the Banach fixed point theorem.

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1 Introduction

The existence of solution to evolution equations with nonlocal conditions in Banach space was studied first by Byszewski [6]. In that paper, he established the existence and uniqueness of mild, strong and classical solutions of the nonlocal Cauchy problem

$$u'(t) + Au(t) = f(t, u(t)), t \in (0, a] \tag{1.1}$$

$$u(0) + g(t_1, t_2, \dots, t_p, u(t_1), u(t_2) \dots, u(t_p) = u_0, \tag{1.2}$$

where $0 < t_1 < \cdots < t_p \leq a$, $-A$ is the infinitesimal generator of a C_0-semigroup in a Banach space X, $u_0 \in X$ and $f : [0, a] \times X \to X$, $g : [0, a]^p \times X^p \to X$ are given functions. The symbol $g(t_1, \dots, t_p, u(\cdot))$ is used in the sense that in the place of "$\cdot$" we can substitute only elements of the set $(t_1, \dots, t_p)$. For example

$$g(t_1, \dots, t_p, u(\cdot)) = C_1 u(t_1) + \cdots + C_p u(t_p),$$

where C_i $(i = 1, 2 \dots, p)$ are given constants. Subsequently he extended the work to various kind of nonlinear evolution equations [1, 5, 6].

Several authors have studied the existence of solutions of abstract quasilinear evolution equations in Banach space [1, 2, 7, 8, 9, 12, 13]. Oka [10] and Oka and Tanaka [11] discussed the existence of solutions of quasilinear integrodifferential equations in Banach spaces. Kato [7] studied the nonhomogeneous evolution equations and Balachandran and Chandrasekaran [2] proved the existence of solution of a delay differential equation with nonlocal condition. An equation of this type occurs in a nonlinear conversation law with memory

$$u(t,x) + \Psi(u(t,x))_x = \int_0^t b(t-s)\Psi(u(t,x))_x \, ds + f(t,x), \quad t \in [0,a], \tag{1.3}$$

$$u(0,x) = \Phi(x), \quad x \in \mathbb{R}. \tag{1.4}$$

It is clear that if nonlocal condition (1.2) is introduced to (1.3), then it will also have better effect than the classical condition $u(0,x) = \Phi(x)$. Therefore, we would like to extend the results for (1.1)-(1.2) to a class of differential equations in Banach spaces.

The aim of this paper is to prove the existence and uniqueness of mild and classical solutions of quasilinear differential equation with nonlocal conditions of the form

$$u'(t) + A(t,u)u(t) = f(t,u(t),u(\alpha(t))), \tag{1.5}$$

$$u(0) + g(u) = u_0, \tag{1.6}$$

where $t \in [0,a]$, $A(t,u)$ is the infinitesimal generator of a C_0-semigroup in a Banach space X, $u_0 \in X$, $f : I \times X \times X \to X$, $g : \mathcal{C}(I : X) \to X$, $\alpha \to I$ are given functions. Here $I = [0,a]$. The results obtained in this paper are generalizations of the results given by Pazy [12], Kato [8, 9] and Balachandran and Ilamaran [3].

2 Preliminaries

Let X and Y be two Banach spaces such that Y is densely and continuously embedded in X. For any Banach spaces Z the norm of Z is denoted by $\|\cdot\|$ or $\|\cdot\|_Z$. The space of all bounded linear operators from X to Y is denoted by $B(X,Y)$ and $B(X,X)$ is written as $B(X)$. We recall some definitions and known facts from Pazy [12].

Definition 2.1 Let S be a linear operator in X and let Y be a subspace of X. The operator $\tilde{S}$ defined by $D(\tilde{S}) = \{x \in D(S) \cap Y : Sx \in Y\}$ and $\tilde{S}x = Sx$ for $x \in D(\tilde{S})$ is called the part of S in Y.

Definition 2.2 Let B be a subset of X and for every $0 \le t \le a$ and $b \in B$, let $A(t,b)$ be the infinitesimal generator of a C_0- semigroup $S_{t,b}(s), s \ge 0$, on X. The family of operators $\{A(t,b)\}, (t,b) \in I \times B$, is stable if there are constants $M \ge 1$ and ω such that

$$\rho(A(t,b)) \supset (\omega,\infty) \quad \text{for } (t,b) \in I \times B,$$

$$\|\prod_{j=1}^{k} R(\lambda : A(t_j,b_j))\| \le M(\lambda-\omega)^{-k}$$

for $\lambda > \omega$ every finite sequences $0 \leq t_1 \leq t_2 \leq \cdots \leq t_k \leq a$, $b_j \in B,\ 1 \leq j \leq k$. The stability of $\{A(t,b)\}, (t,b) \in I \times B$ implies (see [12]) that

$$\|\prod_{j=1}^{k} S_{t_j,b_j}(s_j)\| \leq M \exp\{\omega \sum_{j=1}^{k} s_j\}, \quad s_j \geq 0$$

and any finite sequences $0 \leq t_1 \leq t_2 \leq \cdots \leq t_k \leq a$, $b_j \in B$, $1 \leq j \leq k$. $k = 1, 2, \ldots$

Definition 2.3 Let $S_{t,b}(s), s \geq 0$ be the C_0-semigroup generatated by $A(t,b)$, $(t,b) \in I \times B$. A subspace Y of X is called $A(t,b)$-admissible if Y is invariant subspace of $S_{t,b}(s)$ and the restriction of $S_{t,b}(s)$ to Y is a C_0-semigroup in Y.

Let $B \subset X$ be a subset of X such that for every $(t,b) \in I \times B$, $A(t,b)$ is the infinitesimal generator of a C_0-semigroup $S_{t,b}(s), s \geq 0$ on X. We make the following assumptions:

(E1) The family $\{A(t,b)\}, (t,b) \in I \times B$ is stable.

(E2) Y is $A(t,b)$-admissible for $(t,b) \in I \times B$ and the family $\{\tilde{A}(t,b)\}, (t,b) \in I \times B$ of parts $\tilde{A}(t,b)$ of $A(t,b)$ in Y, is stable in Y.

(E3) For $(t,b) \in I \times B$, $D(A(t,b)) \supset Y$, $A(t,b)$ is a bounded linear operator from Y to X and $t \to A(t,b)$ is continuous in the $B(Y,X)$ norm $\|.\|$ for every $b \in B$.

(E4) There is a constant $L > 0$ such that

$$\|A(t,b_1) - A(t,b_2)\|_{Y \to X} \ \leq L\|b_1 - b_2\|_X$$

holds for every $b_1, b_2 \in B$ and $0 \leq t \leq a$.

Let B be a subset of X and $\{A(t,b)\}, (t,b) \in I \times B$ be a family of operators satisfying the conditions (E1)-(E4). If $u \in \mathcal{C}(I : X)$ has values in B then there is a unique evolution system $U(t,s;u), 0 \leq s \leq t \leq a$, in X satisfying, (see [12, Theorem 5.3.1 and Lemma 6.4.2, pp. 135, 201-202]

(i) $\|U(t,s;u)\| \leq Me^{\omega(t-s)}$ for $0 \leq s \leq t \leq a$. where M and ω are stability constants.

(ii) $\frac{\partial^+}{\partial t}U(t,s;u)w = A(s,u(s))U(t,s;u)w$ for $w \in Y$, for $0 \leq s \leq t \leq a$.

(iii) $\frac{\partial}{\partial s}U(t,s;u)w = -U(t,s;u)A(s,u(s))w$ for $w \in Y$, for $0 \leq s \leq t \leq a$.

(E5) For every $u \in \mathcal{C}(I : X)$ satisfying $u(t) \in B$ for $0 \leq t \leq a$, we have

$$U(t,s;u)Y \subset Y, \quad 0 \leq s \leq t \leq a$$

and $U(t,s;u)$ is strongly continuous in Y for $0 \leq s \leq t \leq a$.

(E6) Y is reflexive.

(E7) For every $(t, b_1, b_2) \in I \times B \times B$, $f(t, b_1, b_2) \in Y$.

Further we assume that

(E8) $g : C(I : B) \to Y$ is Lipschitz continuous in X and bounded in Y, that is, there exist constants $G_0 > 0$ and $G_L > 0$ such that

$$\|g(u)\|_Y \leq G_0,$$
$$\|g(u) - g(v)\|_Y \leq G_L \max_{t \in I} \|u(t) - v(t)\|_X.$$

For the conditions (E9) let Z be taken as both X and Y.

(E9) $f : I \times Z \times Z \to Z$ is continuous and there exist constants $F_L > 0$ and $F_0 > 0$ such that

$$\|f(t, u_1, v_1) - f(t, u_2, v_2)\|_Z \leq F_L(\|u_1 - u_2\|_Z + \|v_1 - v_2\|_Z)$$
$$F_0 = \max_{t \in I} \|f(t, 0, 0)\|_Z.$$

Let us take $M_0 = \max\{\|U(t, s; u)\|_{B(Z)}, 0 \leq s \leq t \leq a, u \in B\}$.

(E10) $\alpha : I \to I$ is absolutely continuous and there exists a constants $b > 0$ such that $\alpha'(t) \geq b$ respectively for $t \in I$.

(E12)

$$M_0 \Big[\|u_0\|_Y + G_0 + r[F_L a(1 + 1/b)] + a(F_0)\Big] \leq r$$

and

$$q = \Big[Ka\|u_0\|_Y + G_0 Ka + M_0 G_L + M_0[F_L a(1 + 1/b)]$$
$$+ Ka[r(F_L a(1 + 1/b)] + a(F_0)\Big] < 1.$$

Next we prove the existence of local classical solutions of the quasilinear problem (1.5)–(1.6).

For a mild solution of (1.5)–(1.6) we mean a function $u \in \mathcal{C}(I : X)$ with values in B and $u_0 \in X$ satisfying the integral equation

$$u(t) = U(t, 0; u)u_0 - U(t, 0; u)g(u) + \int_0^t U(t, s; u)[f(s, u(s), u(\alpha(s)))]ds. \tag{2.7}$$

A function $u \in \mathcal{C}(I : X)$ such that $u(t) \in D(A(t, u(t)))$ for $t \in (0, a]$, $u \in \mathcal{C}^1((0, a] : X)$ and satisfies (1.5)–(1.6) in X is called a classical solution of (1.5)–(1.6) on I. Further there exists a constant $K > 0$ such that for every $u, v \in \mathcal{C}(I : X)$ with values in B and every $w \in Y$, we have

$$\|U(t, s; u)w - U(t, s; v)w\| \leq K\|w\|_Y \int_s^t \|u(\tau) - v(\tau)\|d\tau. \tag{2.8}$$

3 Existence Result

Theorem 3.1 *Let $u_0 \in Y$ and let $B = \{u \in X : \|u\|_Y \leq r\}$, $r > 0$. If the assumptions* (E1)–(E12) *are satisfied, then* (1.5)–(1.6) *has a unique classical solution $u \in \mathcal{C}([0,a] : Y) \cap \mathcal{C}^1((0,a] : X)$*

Let S be a nonempty closed subset of $\mathcal{C}([0,a] : X)$ defined by $S = \{u : u \in \mathcal{C}([0,a] : X), \|u(t)\|_Y \leq r \; for \; 0 \leq t \leq a\}$. Consider a mapping $\mathcal{Q}$ on S defined by

$$(\mathcal{Q}u)(t) = U(t,0;u)u_0 - U(t,0;u)g(u) + \int_0^t U(t,s;u)\Big[f(s,u(s),u(\alpha(s)))\Big]ds.$$

We claim that $\mathcal{Q}$ maps S into S. For $u \in S$, we have

$$\begin{aligned}
&\|\mathcal{Q}u(t)\|_Y \\
&= \|U(t,0;u)u_0 - U(t,0;u)g(u) + \int_0^t U(t,s;u)\Big[f(s,u(s),u(\alpha(s)))\Big]ds\| \\
&\leq \|U(t,0;u)u_0\| + \|U(t,0;u)g(u)\| \\
&\quad + \int_0^t \|U(t,s;u)\|\Big[\|f(s,u(s),u(\alpha(s))) - f(s,0,0)\| + \|f(s,0,0)\|\Big]ds.
\end{aligned}$$

Using assumptions (E8)-(E9), we get

$$\begin{aligned}
\|\mathcal{Q}u(t)\|_Y &\leq M_0\|u_0\|_Y + M_0G_0 + \int_0^t M_0\Big[F_L(\|u(s)\| + \|u(\alpha(s))\|) + F_0\Big]ds \\
&\leq M_0\|u_0\|_Y + M_0G_0 + M_0\Big[F_L ar + F_L\int_0^t \|u(\alpha(s))\|(\alpha'(s)/b)ds + F_0 a\Big] \\
&\leq M_0\|u_0\|_Y + M_0G_0 + M_0\Big[F_L ar + (F_L/b)\int_{\alpha(0)}^{\alpha(t)} \|u(s)\|ds + F_0 a\Big] \\
&\leq M_0\Big[\|u_0\|_Y + G_0 + r[F_L a(1+1/b)] + a(F_0)\Big]
\end{aligned}$$

From assumption (E11), one gets $\|\mathcal{Q}u(t)\|_Y \leq r$. Therefore $\mathcal{Q}$ maps S into itself. Moreover, if $u, v \in S$, then

$$\begin{aligned}
&\|\mathcal{Q}u(t) - \mathcal{Q}v(t)\| \\
&\leq \|U(t,0;u)u_0 - U(t,0;v)u_0\| + \|U(t,0;u)g(u) - U(t,0;v)g(v)\| \\
&\quad + \int_0^t \|U(t,s;u)\Big[f(s,u(s),u(\alpha(s)))\Big] - U(t,s;v)\Big[f(s,v(s),v(\alpha(s)))\Big]\|ds \\
&\leq \|U(t,0;u)u_0 - U(t,0;v)u_0\| + \|U(t,0;u)g(u) - U(t,0;v)g(u)\| \\
&\quad - \|U(t,0;v)g(u) - U(t,0;v)g(v)\| + \int_0^t \Big\{\Big\|U(t,s;u)\Big[f(s,u(s),u(\alpha(s)))\Big]
\end{aligned}$$

$$
\begin{aligned}
&- U(t,s;v)\Big[f(s,u(s),u(\alpha(s)))\Big]\| + \|U(t,s;v)\Big[f(s,u(s),u(\alpha(s)))\Big] \\
&- U(t,s;v)\Big[f(s,v(s),v(\alpha(s)))\Big]\|\Big\}ds
\end{aligned}
$$

Using assumptions (E8)-(E9), one can get

$$
\begin{aligned}
&\|\mathcal{Q}u(t) - \mathcal{Q}v(t)\| \\
&\le Ka\|u_0\|_Y \max_{\tau\in I}\|u(\tau)-v(\tau)\| + G_0Ka\max_{\tau\in I}\|u(\tau)-v(\tau)\| + M_0G_L\max_{\tau\in I}\|u(\tau)-v(\tau)\| \\
&\quad + Ka\max_{\tau\in I}\|u(\tau)-v(\tau)\|\Big[F_L\int_0^t\|u(s)\|ds + F_L\int_0^t\|u(\alpha(s))\|(\alpha'(s)/b)ds + F_0a\Big] \\
&\quad + M_0\Big[F_L\int_0^t\|u(s)-v(s)\|ds + F_L\int_0^t\|u(\alpha(s))-v(\alpha(s))\|(\alpha'(s)/b)ds \\
&\le \Big[Ka\|u_0\|_Y + G_0Ka + M_0G_L + M_0[F_La(1+1/b)] \\
&\quad + Ka[r(F_La(1+1/b)] + a(F_0)\Big]\max_{\tau\in I}\|u(\tau)-v(\tau)\| \\
&= q\max_{\tau\in I}\|u(\tau)-v(\tau)\|
\end{aligned}
$$

where $0 < q < 1$. From this inequality it follows that for any $t \in I$,

$$
\|\mathcal{Q}u(t) - \mathcal{Q}v(t)\| \le q\max_{\tau\in I}\|u(\tau)-v(\tau)\|,
$$

so that $\mathcal{Q}$ is a contraction on S. From the contraction mapping theorem it follows that $\mathcal{Q}$ has a unique fixed point $u \in S$ which is the mild solution of (1.5)–(1.6)on $[0,a]$. Note that $u(t)$ is in $\mathcal{C}(I:Y)$ by (E6) see [12, pp. 135, 201-202 lemma 7.4]. In fact , $u(t)$ is weakly continuous as a Y-valued function. This implies that $u(t)$ is separably valued in Y, hence it is strongly measurable. Then $\|u(t)\|_Y$ is bounded and measurable function in t. Therefore, $u(t)$ is Bochner integrable (see e.g. [14, Chap.V]). Using relation $u(t) = \mathcal{Q}u(t)$, we conclude that $u(t)$ is in $\mathcal{C}(I:Y)$.

Now consider the evolution equation

$$
v'(t) + B(t)v(t) = h(t), \quad t \in [0,a] \tag{3.9}
$$

$$
v(0) = u_0 - g(u) \tag{3.10}
$$

where $B(t) = A(t,u(t))$ and $h(t) = f(t,u(t),u(\alpha(t)))$, $t \in [0,a]$ and u is the unique fixed point of $\mathcal{Q}$ in S. We note that $B(t)$ satisfies (H1)-(H3) in [12, Sec. 5.5.3] and $h \in \mathcal{C}(I:Y)$. [12, Theorem 5.5.2] implies that there exists a unique function $v \in \mathcal{C}(I:Y)$ such that $v \in \mathcal{C}^1((0,a],X)$ satisfying (3.9) and (3.10) in X and v is given by

$$
v(t) = U(t,0;u)u_0 - U(t,0;u)g(u) + \int_0^t U(t,s;u)[f(s,u(s),u(\alpha(s)))]ds,
$$

where $U(t,s;u)$ is the evolution system generated by the family $\{A(t,u(t))\}, t \in I$ of the linear operators in X. The uniqueness of v implies that $v = u$ on I and hence u is a unique classical solution of (1.5))–(1.6) and $u \in \mathcal{C}([0, a] : Y) \cap \mathcal{C}^1((0, a] : X)$.

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