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3 authors:

Paul Samuel Francis
University of Eastern Africa, Baraton

Dr. Gunasekar T
Vel Tech Rangarajan Dr.Sagunthala R&D Institute of Science and Technology, Ava…

M. Mallika Arjunan
SASTRA University

CONTROLLABILITY RESULTS FOR SECOND ORDER IMPULSIVE NEUTRAL FUNCTIONAL INTEGRODIFFERENTIAL INCLUSIONS WITH INFINITE DELAY

F. Paul Samuel[1]

Department of Mathematics and Physics
University of Eastern Africa
Baraton
Eldoret 2500 - 30100
Kenya
e-mail: paulsamuel_f@yahoo.com

T. Gunasekar

Department of Mathematics, Karunya University
Coimbatore - 641 114
India
Present Address:
School of Advanced Sciences
Fluid Dynamics Division
VIT University
Vellore - 632 014
India
e-mail: gunasekar.t@vit.ac.in

M. Mallika Arjunan

Department of Mathematics
C.B.M. College
Kovaipudur
Coimbatore - 641 042
India
e-mail: arjunphd07@yahoo.co.in

Abstract. This paper is concerned with the controllability of a partial neutral functional integrodifferential inclusion of second order with impulse effect and infinite delay in Banach spaces. The controllability of mild solutions using a fixed point theorem for contraction multi-valued maps and without assuming compactness of the family of cosine operators.

Keywords: Controllability; Second Order Abstract Cauchy problem; Impulsive systems; Phase space; Cosine function of operators; Integrodifferential inclusions.
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[1]Corresponding author.

1. Introduction

The controllability for functional differential and integrodifferential systems in Banach spaces has found wide applications in many branches of physics and technical sciences. The theory of impulsive integrodifferential equations is an important branch of differential equations, which has an extensively physical background [14], [16], [22]. Delay differential equations are a special type of functional differential and integrodifferential equations which is similar to the ordinary differential equation, but the evolution involves past values of the state variable. The solutions of delay differential equations therefore requires knowledge of not only the current state, but also of the state of a certain time previously. System of delay differential equations have occurred all areas of science and particularly in the biological sciences (e.g., population dynamics and epidemiology). They also often arise in either natural or technological control problems. In these systems, a controller monitors the state of the system, and makes adjustments to the system based on its observations. Since these adjustments can never be made instantaneously, a delay arises between the observation and the control action. There are different kinds of delay existing in dynamical systems. The development of the theory of functional differential equations with infinite delay heavily depends on a choice of a phase space. In fact, various phase spaces have been considered and each different phase space has required a separate development of the theory. When the delay is infinite, the selection of the state(i.e. phase space) plays an important role in the study of both qualitative and quantitative theory. A usual choice is a normed space satisfying suitable axioms, which was introduced by Hale and Kato [15]. Hernandez et al. [16] proved the existence of solutions for impulsive partial neutral functional differential equations for first and second order systems with infinite delay. Balachandran et al. [1], [2], [3] discussed the controllability first and second order functional differential and integrodifferential evolution system in a Banach space without impulse effect. Chang and Li [9] obtained the controllability result for functional integrodifferential inclusions on an unbounded domain without impulse term. In recent years, significant progress has been made in the controllability of linear and nonlinear deterministic systems [19], [20], [21]

On the other hand, researchers have been proving the controllability results using compactness assumption of semigroups and the family of cosine operators. Bing Liu [23] studied the controllability of first order impulsive neutral functional intrgrodifferential inclusions with infinite delay in a Banach space with the assumption of compactness of the semigroup. However, as remarked by Triggiani [24], in an infinite dimensional Banach space, the linear control system is never exactly controllable on a given interval of time, if either a bounded linear operator (from control space to state space) is compact or a semigroup is compact. According to Triggiani [24], this is a typical case for most control systems governed by parabolic partial differential equations, and hence the concept of exact controllability is very limited for many parabolic partial differential equations. Nowadays, researchers are engaged to overcome this problem, refer to [15], [16], [17], [23], [24]. Recently, Chalishajar and Gunasekar et al. [9], [6], [7], [8], [14] studied the controllability of second order neutral functional differential inclusion, with infinite delay and impulse effect on unbounded domain, without compactness of the family of cosine operators.

In this paper, we discuss the controllability for the second order impulsive neutral functional integrodifferential inclusions, with infinite delay through the phase space defined in [17], and without compactness of the family of cosine operators. To the best of our knowledge, there is no work reported on the controllability of neutral functional integrodifferential inclusion of second order with impulse effect and infinite delay in a Phase space, and the aim of this paper is to close the gap. The result obtained in this paper are generalizations of the results given by Chalishajar [8] and Covitz and Nadler [11].

2. Preliminaries

The purpose of this paper is to study the controllability of impulsive partial neutral functional integrodifferential inclusion of second order with infinite delay. Specifically, we are concerned with the inclusion form

$$(1)\qquad \begin{cases} \dfrac{d}{dt}\left[x'(t) - g(t, x_t, x'(t))\right] \in Ax(t) + Bu(t) \\ \qquad +F\Big(t, x_t, x'(t), \displaystyle\int_0^t e(t, s, x_s, x'(s))ds\Big), \ t \in J, \ t \neq t_k; \\ \Delta x(t_k) = I_k^1(x(t_k), x'(t_k)); \ k = 1, 2, ..., m \\ \Delta x(t_k) = I_k^2(x(t_k), x'(t_k)); \ k = 1, 2, ..., m \\ x(0) = \phi \in \mathcal{B}_h, \quad x'(0) = x_0, \end{cases}$$

where $A : D(A) \subset X \to X$ is the infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in R\}$ defined on X, and $F : J \times \mathcal{B}_h \times X \times X \Rightarrow X$ is bounded, closed and convex multivalued map, $h : J \times J \times \mathcal{B}_h \times X \Rightarrow X$ is appropriate function. Let $J_0 = (-\infty, 0]$, and non-local condition $\phi \in \mathcal{B}_h$ (defined below) and $x_0 \in X$ be the given initial values. $g : J \times \mathcal{B}_h \times X \to X$ is a given function, the state the function $x(t)$ takes values in X, and the control function $u \in L^2(J, U)$, a Banach space of admissible control functions with U as a Banach space. B is a bounded linear operator U to X. X is a Banach space with the norm $|\cdot|$.

Also, $0 < t_0 < t_1 < ... < t_p < t_{p+1} = m \quad (\to \infty \quad as \quad t \to \infty)$; $I_k^1, I_k^2 \in C(X \times X, X)$, $k = 1, 2, ..., p$ are bounded, $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-), \Delta x'|_{t=t_k} = x'(t_k^+) - x'(t_k^-)$ and $x(t_k^-), x(t_k^+), x'(t_k^-), x'(t_k^+)$ represents the left and right limits of $x(t)$ and $x'(t)$, respectively, at $t = t_k$. Furthermore, for any continuous function x defined on the interval $J_1 = (-\infty, m]$ with values in X and for any $t \in J$. We denote by x_t an element of $C(J_0, X)$ defined by $x_t(\theta) = x(t + \theta), \theta \in J_0$.

We present the abstract phase space $\mathcal{B}_h$. Assume that $h : (-\infty, 0] \to [0, \infty)$ be a continuous function with $l = \int_{-\infty}^{0} h(s)ds < +\infty$. Define

$$\mathcal{B}_h := \Big\{ \phi :]-\infty, 0] \to X \text{ such that, for any } r > 0, \\ \phi(\theta) \text{ is bounded and measurable function on } [-r, 0] \\ \text{and } \int_{-\infty}^{0} h(s) \sup_{s \leq \theta \leq 0} |\phi(\theta)| ds < +\infty \Big\}.$$

Here, $\mathcal{B}_h$ is endowed with the norm

$$\|\phi\|_{\mathcal{B}_h} = \int_{-\infty}^{0} h(s) \sup_{s \leq \theta \leq 0} |\phi(\theta)| ds, \qquad \forall \phi \in \mathcal{B}_h.$$

Then, it is easy to show that $(\mathcal{B}_h, \|.\|_{\mathcal{B}_h})$ is a Banach space.

Lemma 2.1 [8] *Suppose $x \in \mathcal{B}_h$; then, for each $t \in J, x_t \in \mathcal{B}_h$. Moreover,*

$$l|x(t)| \leq \|x_t\|_{\mathcal{B}_h} \leq \sup_{s \leq \theta \leq 0} (|x(s) + \|x_0\|_{\mathcal{B}_h}),$$

where $l := \displaystyle\int_{-\infty}^{0} h(s)ds < +\infty$.

Next, we introduce definitions, notation and preliminary facts from multi-valued analysis, which are useful for the development of this paper. Let $C(J, X)$ be the Banach space of continuous functions from J to X with the norm

$$\|y\|_{\infty} = \sup_{t \in J} |y(t)|.$$

$B(X)$ denotes the Banach space of bounded linear operators from X to X.

Let $L^1(J, X)$ denotes the Banach space of continuous function $x : J \to X$, which are integrable and endowed with the norm

$$\|x\|_{L^1} = \int_0^m |x(t)|dt, \; x \in L^1(J, X).$$

For a metric space (X, d), we introduce the following notations:

$$P(X) := \{Y \in \mathcal{P}(X) : Y \neq \phi\},$$
$$P_{cl}(X) := \{Y \in P(X) : Y \text{is closed}\},$$
$$P_b(X) := \{Y \in P(X) : Y \text{is bounded}\},$$
$$P_{cp}(X) := \{Y \in P(X) : Y \text{is compact}\},$$
$$P_{b,cl}(X) := \{Y \in P(X) : Y \text{is bounded and closed}\}.$$

We define a Hausdorff space $H_d : P(X) \times P(X) \to \mathbb{R}^+ \cup \{\infty\}$ by

$$H_d(A, B) := \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b) \right\},$$

where $d(A, b) := \inf_{a \in A} d(a, b), \; d(a, B) := \inf_{b \in B} d(a, b)$.

Then $(P_{b,cl}(X), H_d)$ is a metric space, and $(P_{cl}(X), H_d)$ is a generalized (complete) metric space.

We now recall some preliminaries about multi-valued maps.

Let $(X, \|.\|)$ be a Banach space. A multi-valued map $G : X => X$ is a convex (resp. closed) iff $G(y)$ is convex (resp. closed) in X, for all $y \in X$. The map G is bounded on bounded sets iff $G(B) = \bigcup_{y \in B} G(y)$ is bounded in X for any bounded set B on X (i.e., $\sup_{y \in B}\{\sup \|x\| : x \in G(y)\}\} < \infty$), G is called upper semi-continuous (u.s.c.) on X iff for each $y_0 \in X$ the set $G(y_0)$ is a nonempty, closed subset of X and if for each open set B of X containing $G(y_0)$ there exists an open neighbourhood A of y_0 such that $G(A) \subseteq B$. The map G is said to be completely continuous iff $G(B)$ is relatively compact for every bounded subset $B \subseteq X$.

If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. iff G has a closed graph. That is, if $y_n \to y_0$ and $x_n \to x_0$, where $x_n \in G(y_n)$, then $x_0 \in G(y_0)$. We say that G has a fixed point iff there is $y \in X$ such that $y \in G(y)$. In the following, $BCC(X)$ denotes the set of all nonempty bounded, closed and convex subsets of X.

A multivalued map $G : J \to BCC(X)$ is said to be measurable iff for each $y \in X$, the distance function $Y : J \to \mathbb{R}$, defined by

$$Y(t) := d(y, G(t)) = \inf\{|y - z| : z \in G(t)\},$$

is measurable. For more details on multivalued maps see [12], [18].

An upper semi-continuous map $G : X => X$ is said to be condensing iff for any subset $B \subseteq X$, with $\alpha(B) \neq 0$ we have $\alpha(G(B)) < \alpha(B)$, where α denotes the Kuratowski measure of non-compactness. For the properties of the Kuratowski measure, we refer to Banas and Geobel [4].

We note that completely continuous multivalued map is the easiest example of a condensing map. For more details on multivalued maps, see the book of Deiming [12] and the research article of Travis and Webb [25].

We say that the family $\{C(t) : t \in \mathbb{R}\}$ of operators in $B(X)$ is a strongly continuous cosine family iff

1. $C(0) = I$, I is the identity operator on X.
2. $C(t + s) + C(t - s) = 2C(t)C(s)$ for all $s, t \in \mathbb{R}$.
3. The map $t \mapsto C(t)y$ is strongly continuous for each $y \in X$.

The strongly continuous sine family $\{S(t) : t \in \mathbb{R}\}$, associated with the strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$, is defined by

$$S(t)y := \int_0^t C(s)yds, y \in X, t \in \mathbb{R}.$$

The infinitesimal generator $A : X \to X$ of a cosine family $C(t) : t \in \mathbb{R}$ is defined by

$$Ay := \frac{d^2}{dt^2}\, C(t)y|_{t=0}, y \in D(A).$$

where $D(A) = \{y \in X : C(t)y \ \text{ is twice continuous differentiable}\}$.

We refer to the book of Goldstein [13] for the detailed study of the family of cosine and sine operators.

Definition 2.1 A multivalued operator $G : C => P_{cl}(X)$ is called

(a) γ-Lipschitz iff there exists $\gamma > 0$ such that $H_d(G(x), G(y)) \le \gamma d(x, y)$ for each $x, y \in X$;

(b) a contraction iff it is γ-Lipschitz with $\gamma < 1$.

Definition 2.2 The integral formulation $x(t)$ of system (1) is given by

$$\begin{aligned} x(t) &:= \phi(t), \quad \text{if } t \in J_0, \\ x(t) &:= C(t)\phi(0) + S(t)[x_0 - g(0, \phi, x_0)] + \int_0^t C(t-s)g(s, x_s, x'(s))ds \\ &\quad + \int_0^t S(t-s)Bu(s)ds + \int_0^t S(t-s)f(s)ds \\ &\quad + \sum_{0<t_k<t} C(t-t_k)I_k^1(x_{t_k}, x'(t_k)) + \sum_{0<t_k<t} S(t-t_k)I_k^2(x_{t_k}, x'(t_k)), \ \text{if } t \in J, \end{aligned}$$

where $f \in S^1_{F,z,y} := \{f \in L^1(J, X) : f(t) \in F(t, x_t, x'(t), \int_0^t e(t, s, x_s, x'(s))ds)$, for a.e. $t \in J\} \neq \phi$ and $x_0 = \phi \in B_h$ is called a mild solution of the inclusion (1) provided $\int_0^t C(t-s)g(s, x_s, x'(s))ds$ is integrable.

Definition 2.3 System (1) is said to be controllable on $J' = J \mid \{t_1, t_2, ..., t_p\}$ iff for every $\phi \in \mathcal{B}_h$ with $\phi(0) = D(A), x_0 \in X$, and $x_1 \in X$, there exists a control $u \in L^2(J, U)$ such that the mild solution $x(\cdot)$ of the system (1) satisfies $x(m) = x_1 \in D(A), x'(m) = y_1 \in X$, the conditions $\Delta x|_{t=t_k} = I_k^1(x_{t_k}, x'(t_k))$, and $\Delta x'|_{t=t_k} = I_k^2(x_{t_k}, x'(t_k)); k = 1, 2, ..., p$; and $x'(0) = x_0 \in \mathcal{B}_h$.

Lemma 2.2 ([25]) *Let $\{C(t) : t \in R\}$ be a strongly continuous cosine family in X with infinitesimal generator A. If $h_1 : R \to X$ is continuously differentiable, with $z_0 \in D(A), z_1 \in X$, and*

$$w(t) = C(t)z_0 + S(t)z_1 + \int_0^t S(t-s)h(s)ds, \quad t \in R,$$

then $w(t) \in D(A)$ for $t \in R$, w is twice continuously differentiable, and w satisfies

$$w''(t) = Aw(t) + h_1(t), \quad t \in R, \quad w(0) = x_1, \quad w'(0) = y_0.$$

Conversely, if $h_1 : R \to X$ is continuous, $w(t) : R \to X$ is twice continuously differentiable, $w(t) \in D(A)$ for $t \in R$, and w satisfies system (1). *Then*

$$w(t) = C(t)z_0 + S(t)z_1 + \int_0^t S(t-s)h(s)ds, \quad t \in R.$$

Lemma 2.3 (Covitz and Nadler [11]) *Let (X, d) be a complete metric space. If $G : X \Rightarrow P_{cl}(X)$ is a contraction, then fix $G = \phi$ (where fix G denotes set of fixed points of the multi-valued operator G).*

3. Controllability results

In this section, we prove the controllability of system (1). We adopt the following hypotheses:

(H1) An operator A is an infinitesimal generator of a strongly continuous and bounded cosine family $\{C(t) : t \in R\}$ with $M = \sup\{|C(t)| : t \in J\}$, and sine family $\{S(t) : t \in R\}$ with $\widetilde{M} = \sup\{|S(t)| : t \in J\}$.

(H2) $F : J \times \mathcal{B}_h \times X \times X \Rightarrow X : P_{cp}(X) : (., \psi, x, w) \Rightarrow F(., \psi, x, w)$ is measurable with respect to t for each $\psi \in \mathcal{B}_h$ and $x, w \in X$, and F is u.s.c. with respect to second,third and fourth variables, for each $t \in J$.

Indeed, by the Caratheodory condition of F, F has a measurable selection (see Theorem III in [5]).

(H3) The linear operator $W : L^2(J, U) \to X$ is defined by

$$Wu := \int_0^b S(t-s)Bu(s)ds,$$

W has an invertible operator W^{-1} which take the values in $L^2(J, U)|\ker W$ (since the restriction of W to the domain $L^2(J, U)|\ker W$ is invertible) and there exists positive constants M_1 such that $\|W^{-1}\| \leq M_1$.

(H4) The linear operator $\overline{W} : L^2(J, U) \to X$ is defined by

$$\overline{W}u := \int_0^b C(t-s)Bu(s)ds,$$

$\overline{W}$ has an invertible operator $\overline{W}^{-1}$ which take the values in $L^2(J, U)|\ker \overline{W}$ and there exists positive constants M_2 such that $\|\overline{W}^{-1}\| \leq M_2$.

(H5) There exist a positive constant M_3 such that $\|B\| \leq M_3$ for $t \in J$.

(H6) $H_d(F(t, \psi_1, u_1, w_1), F(t, \psi_2, u_2, w_2)) \leq l(t)(\|\psi_1 - \psi_2\| + |u_1 - u_2| + |w_1 - w_2|)$, for each $t \in J$, $\psi_1, \psi_2 \in \mathcal{B}_h$ and $u_1, u_2, w_1, w_2 \in X$; where $l \in L^1(J, R^+)$ and $d(0, F(t, 0, 0, 0)) \leq l(t)$, for a.e. $t \in J$.

(H7) The function $e : J \times J \times \mathcal{B}_h \times X \Rightarrow X$ satisfies the following conditions:

(i) For every $(t, s) \in J \times J$ the function $h(t, s, .) : \mathcal{B}_h \times X \to X$ is continuous.

(ii) For every $(\psi, w) \in \mathcal{B}_h \times X$ the function $e(., \psi, w) : J \times J \to X$ is strongly measurable.

(iii) $H_d(e(t,s,\psi_1,u_1), e(t,s,\psi_2,u_2)) \le l'(t)(\|\psi_1-\psi_2\| + |u_1-u_2|)$, for each $t \in J$, $\psi_1,\psi_2 \in \mathcal{B}_h$ and $u_1,u_2 \in X$; where $l' \in L^1(J,R^+)$ and $d(0, e(t,s,0,0)) \le l'(t)$, for a.e. $t \in J$.

(H8) (i) $I_k^1 : \mathcal{B}_h \times X \to X$ are completely continuous and there exist constants $c_k^j, j = 1,2$, such that $\|I_k^1(\psi,x)\|_X \le (c_k^1(\|\psi\|\mathcal{B}_h + |x|)) + c_k^2$; $k = 1,2,...,p$; for every $(\psi,x) \in \mathcal{B}_h \times X$.

(ii) $I_k^2 : \mathcal{B}_h \times X \to X$ are completely continuous and there exist constants $d_k^j, j = 1,2$, such that $\|I_k^2(\psi,x)\|_X \le (d_k^1(\|\psi\|\mathcal{B}_h + |x|)) + d_k^2$; $k = 1,2,...,p$; for every $(\psi,x) \in \mathcal{B}_h \times X$.

Theorem 3.1 *Assume that the hypotheses* (H1)-(H8) *be satisfied. System* (1) *is controllable for* x_0 *and* x_1 *on* J*, provided* $m\widetilde{M}L[1+(1/2)M_3m(M_1\widetilde{M}+M_2M)] < 1$.

Proof. For $k = 1,2,...,p$, consider $\mathcal{B}_b(J_1,X) = \{x :]-\infty,m] \to X;\ x(t)$ is continuous at $t \neq t_k$, $x(t_k^-) = x(t_k)$, and $x(t_k^+)$ exists$\}$, and $\mathcal{B}_b^1(J_1,X) = \{x \in \mathcal{B}_b(J_1,X);\ x'(t)$ is continuous at $t \neq t_k$, $x'(t_k^-) = x'(t_k)$, and $x'(t_k^+)$ exists$\}$, with the norm

$$\|x\|_{b^1} = \|x\|_{\mathcal{B}_h} + \sup_{t\in J_m}\{\|x(s)\|_b, \|x'(s)\|_{b^1}\} : 0 \le s \le b,\ x \in \mathcal{B} \text{ and } x' \in \mathcal{B}_b^1;$$

where $\|x\|_b = \sup_{t\in J_m} |x(t)|$. Here, x_k is restriction of x to $J_k = (t_k, t_{k+1}]$; such that

$$\|x_k\|_{J_k} = \sup_{s\in J_k} \|x_k(s)\|.$$

Let us define the spaces

$$Z_1 := \{\mathcal{B}_b(J_1,X) \cap C^2(J_1',X)\}, \quad Z_2 := \{\mathcal{B}_b^1(J_1,X) \cap C^2(J_1',X)\},$$

where $J_1' = J_1|\{t_k, k = 1,2,...,p\}$.

Using hypotheses (H3) and (H4), we defined the control formally as

$$\begin{aligned} u_y(t) = \frac{1}{2}\Big\{ & W^{-1}\Big[x_1 - C(m)\phi(0) - S(m)[x(0) - g(0,\phi,x_0)] \\ & - \int_0^m C(m-s)g(s,x_s,x'(s))ds - \int_0^m S(m-s)f(s)ds \\ & - \sum_{0<t_k<t} C(m-t_k)I_k^1(x_{t_k},x'(t_k)) - \sum_{0<t_k<t} S(m-t_k)I_k^2(x_{t_k},x'(t_k))\Big] \\ & + \overline{W}^{-1}\Big[y_1 - AS(m)\phi(0) - C(m)[x(0) - g(0,\phi,x_0)] \\ & - \int_0^m AS(m-s)g(s,x_s,x'(s))ds - \int_0^m C(m-s)f(s)ds \\ & - \sum_{0<t_k<t} AS(m-t_k)I_k^1(x_{t_k},x'(t_k)) - \sum_{0<t_k<t} C(m-t_k)I_k^2(x_{t_k},x'(t_k))\Big\}(t), \end{aligned}$$

where $v \in S^1_{F,z,x}$.

Using this control, we shall now show that the operator $G_1 : Z_1 \Rightarrow \mathcal{P}(Z_1)$ and $G_2 : Z_2 \Rightarrow \mathcal{P}(Z_2)$, defined by

$$\begin{aligned} G_1(y) := \Big\{ & x \in Z_1 : x(t) = C(t)\phi(0) + S(t)[x_0 - g(0,\phi,x_0)] \\ & + \int_0^t C(t-s)g(s,x_s,x'(s))ds + \int_0^t S(t-s)f(s)ds + \int_0^t S(t-s)Bu_y(s)ds \\ & + \sum_{0<t_k<t} C(t-t_k)I_k^1(x_{t_k},x'(t_k)) + \sum_{0<t_k<t} S(t-t_k)I_k^2(x_{t_k},x'(t_k)) \Big\}, \end{aligned}$$

$$\begin{aligned} G_2(y') := \Big\{ & x' \in Z_1 : x(t) = AS(t)\phi(0) + C(t)[x_0 - g(0,\phi,x_0)] \\ & + \int_0^t AS(t-s)g(s,x_s,x'(s))ds + \int_0^t C(t-s)f(s)ds + \int_0^t C(t-s)Bu_y(s)ds \\ & + \sum_{0<t_k<t} AS(t-t_k)I_k^1(x_{t_k},x'(t_k)) + \sum_{0<t_k<t} C(t-t_k)I_k^2(x_{t_k},x'(t_k)) \Big\}, \end{aligned}$$

has a fixed point. This fixed point is then a mild solution of system (1). Obviously, $x_1 \in (G_1 y)(m)$ and $x_2 \in (G_2 y')(m)$. Next, we shall show that G_1 and G_2 satisfy the assumptions of Lemma (2.3). The proof will be given in two steps.

Step 1. We show that $G_1(y) \in P_{cl}(Z_1)$, $\quad G_2(y) \in P_{cl}(Z_2)$.
Indeed, let $\{x_n\}_{n\geq 0} \in G_1(y)$ such that $x_n \to x_*$ in Z_1. Then $x_* \in Z_1$, and there exists $f_n \in S^1_{F,z,x}$ such that for each $t \in J$.

$$\begin{aligned} x_n(t) = {} & C(t)\phi(0) + S(t)[x_0 - g(0,\phi,x_0)] \\ & + \int_0^t C(t-s)g(s,x_s,x'(s))ds + \int_0^t S(t-s)f_n(s)ds \\ & + \sum_{0<t_k<t} C(t-t_k)I_k^1(x_{t_k},x'(t_k)) + \sum_{0<t_k<t} S(t-t_k)I_k^2(x_{t_k},x'(t_k)) \\ & + \frac{1}{2}\int_0^t S(t-s)B\Big\{W^{-1}\Big[x_1 - C(m)x_0 - S(m)[x_1 - g(0,\phi,x_0)] \\ & - \int_0^m C(m-\tau)g(\tau,x_\tau,x'(\tau))ds - \int_0^m S(m-\tau)f_n(\tau)d\tau \\ & - \sum_{0<t_k<t} C(m-t_k)I_k^1(x_{t_k},x'(t_k)) - \sum_{0<t_k<t} S(m-t_k)I_k^2(x_{t_k},x'(t_k))\Big] \\ & + \overline{W}^{-1}\Big[y_1 - AS(m)\phi(0) - C(m)[x(0) - g(0,\phi,x_0)] \\ & - \int_0^m AS(m-\tau)g(\tau,x_\tau,x'(\tau))ds - \int_0^m C(m-\tau)f_n(\tau)d\tau \\ & - \sum_{0<t_k<t} AS(m-t_k)I_k^1(x_{t_k},x'(t_k)) - \sum_{0<t_k<t} C(m-t_k)I_k^2(x_{t_k},x'(t_k))\Big]\Big\}(s)ds \end{aligned}$$

Using the fact that F has compact values and (H6) holds, we may pass a subsequence which is necessary to find that f_n converges to f in $L^1(J,X)$; hence $f \in S^1_{F,z,x}$.

Then, for each $t \in J$,

$$
\begin{aligned}
& x_n(t) \to x_*(t) = C(t)\phi(0) + S(t)[x_0 - g(0,\phi,x_0)] \\
& + \int_0^t C(t-s)g(s,x_s,x'(s))ds + \int_0^t S(t-s)f_n(s)ds \\
& + \sum_{0<t_k<t} C(t-t_k)I_k^1(x_{t_k},x'(t_k)) + \sum_{0<t_k<t} S(t-t_k)I_k^2(x_{t_k},x'(t_k)) \\
& +\frac{1}{2}\int_0^t S(t-s)B\Big\{W^{-1}\Big[x_1 - C(m)x_0 - S(m)[x_1 - g(0,\phi,x_0)] \\
& - \int_0^m C(m-\tau)g(\tau,x_\tau,x'(\tau))ds - \int_0^m S(m-\tau)f_n(\tau)d\tau \\
& - \sum_{0<t_k<t} C(m-t_k)I_k^1(x_{t_k},x'(t_k)) - \sum_{0<t_k<t} S(m-t_k)I_k^2(x_{t_k},x'(t_k))\Big] \\
& +\overline{W}^{-1}\Big[y_1 - AS(m)\phi(0) - C(m)[x(0) - g(0,\phi,x_0)] \\
& - \int_0^m AS(m-\tau)g(\tau,x_\tau,x'(\tau))ds - \int_0^m C(m-\tau)f_n(\tau)d\tau \\
& - \sum_{0<t_k<t} AS(m-t_k)I_k^1(x_{t_k},x'(t_k)) - \sum_{0<t_k<t} C(m-t_k)I_k^2(x_{t_k},x'(t_k))\Big]\Big\}(s)ds.
\end{aligned}
$$

So, $x_* \in G_1(y)$. In particular, $G_1(y) \in P_{cl}(Z_1)$. Similarly, we can show that $G_2(y') \in P_{cl}(Z_2)$.

Step 2. We shall show $G_1(y)$ and $G_2(y')$ are contractive multi-valued maps for each $x \in Z_1$ and $x' \in Z_2$.
Let $x_t, \overline{x_t}, x', \overline{x'} \in Z_1$ and $x \in G_1(y)$. then there exists $f \in S^1_{F,z,x}$ such that

$$
\begin{aligned}
x(t) \; & = C(t)\phi(0) + S(t)[x_0 - g(0,\phi,x_0)] \\
& + \int_0^t C(t-s)g(s,x_s,x'(s))ds + \int_0^t S(t-s)f(s)ds \\
& + \sum_{0<t_k<t} C(t-t_k)I_k^1(x_{t_k},x'(t_k)) + \sum_{0<t_k<t} S(t-t_k)I_k^2(x_{t_k},x'(t_k)) \\
& +\frac{1}{2}\int_0^t S(t-s)B\Big\{W^{-1}\Big[x_1 - C(m)x_0 - S(m)[x(0) - g(0,\phi,x_0)] \\
& - \int_0^m C(m-\tau)g(\tau,x_\tau,x'(\tau))ds - \int_0^m S(m-\tau)f(\tau)d\tau \\
& - \sum_{0<t_k<t} C(m-t_k)I_k^1(x_{t_k},x'(t_k)) - \sum_{0<t_k<t} S(m-t_k)I_k^2(x_{t_k},x'(t_k))\Big] \\
& +\overline{W}^{-1}\Big[y_1 - AS(m)\phi(0) - C(m)[x(0) - g(0,\phi,x_0)] \\
& - \int_0^m AS(m-\tau)g(\tau,x_\tau,x'(\tau))ds - \int_0^m C(m-\tau)f(\tau)d\tau \\
& - \sum_{0<t_k<t} AS(m-t_k)I_k^1(x_{t_k},x'(t_k)) - \sum_{0<t_k<t} C(m-t_k)I_k^2(x_{t_k},x'(t_k))\Big]\Big\}(s)ds
\end{aligned}
$$

From (H6),(H7) it follows that, for each $t \in J$.

$$H_d(F(t,x_t,x'(t),w),F(t,\overline{x}_t,\overline{x'}(t),\overline{w})) \le l(t)\Big(\|x_t-\overline{x}(t)\|+|x'(t)-\overline{x'}(t)|+|w-\overline{w}|\Big).$$

Hence, there exists $\overline{f}(t) \in F(t,\overline{x}_t,\overline{x'}(t),\overline{w})$ such that

$$|f(t,x_t,x'(t),w)-\overline{f}(t,\overline{x}_t,\overline{x'}(t),\overline{w})| \le l(t)\Big(\|x_t-\overline{x}(t)\|+|x'(t)-\overline{x'}(t)|+|w-\overline{w}|\Big).$$

Consider $V : J \Rightarrow \mathcal{P}(X)$, given by

$$\begin{aligned} V(t) := \Big\{\overline{f}(t) \in X : |f(t,x_t,x'(t),w)-\overline{f}(t,\overline{x}_t,\overline{x'}(t),\overline{w})| \\ \le l(t)\Big(\|x_t-\overline{x}(t)\|+|x'(t)-\overline{x'}(t)|+|w-\overline{w}|\Big)\Big\}. \end{aligned}$$

Since the multi-valued operator $W(t) = V(t) \cap F(t,\overline{x}_t,\overline{x'}(t),\overline{w})$ is a measurable (see Proposition III.4 in [5]), there exists a function $\overline{f}(t)$ which is measurable selection for W. So, $\overline{f}(t) \in F(t,\overline{x}_t,\overline{x'}(t),\overline{w})$, and

$$|f(t)-\overline{f}(t)| \le l(t)\Big(\|x_t-\overline{x}(t)\|+|x'(t)-\overline{x'}(t)|+|w-\overline{w}|\Big), \text{ for each } t \in J.$$

Furthermore, for each $t \in J$, we define

$$\begin{aligned} \overline{x}(t) &= C(t)\phi(0)+S(t)[x_0-g(0,\phi,x_0)] \\ &+\int_0^t C(t-s)g(s,x_s,x'(s))ds+\int_0^t S(t-s)\overline{f}(s)ds \\ &+\sum_{0<t_k<t} C(t-t_k)I_k^1(x_{t_k},x'(t_k))+\sum_{0<t_k<t} S(t-t_k)I_k^2(x_{t_k},x'(t_k)) \\ &+\frac{1}{2}\int_0^t S(t-s)B\Big\{W^{-1}\Big[x_1-C(m)x_0-S(m)[x(0)-g(0,\phi,x_0)] \\ &-\int_0^m C(m-\tau)g(\tau,x_\tau,x'(\tau))ds-\int_0^m S(m-\tau)\overline{f}(\tau)d\tau \\ &-\sum_{0<t_k<t} C(m-t_k)I_k^1(x_{t_k},x'(t_k))-\sum_{0<t_k<t} S(m-t_k)I_k^2(x_{t_k},x'(t_k))\Big] \\ &+\overline{W}^{-1}\Big[y_1-AS(m)\phi(0)-C(m)[x(0)-g(0,\phi,x_0)] \\ &-\int_0^m AS(m-\tau)g(\tau,x_\tau,x'(\tau))ds-\int_0^m C(m-\tau)\overline{f}(\tau)d\tau \\ &-\sum_{0<t_k<t} AS(m-t_k)I_k^1(x_{t_k},x'(t_k))-\sum_{0<t_k<t} C(m-t_k)I_k^2(x_{t_k},x'(t_k))\Big]\Big\}(s)ds \end{aligned}$$

Then, for each $t \in J$, we get

$$
\begin{aligned}
&|x(t) - \overline{x}(t)| \leq \Big| \int_0^t S(t-s)[f(s) - \overline{f}(s)]ds \Big| \\
&+\frac{1}{2} \Big| \int_0^t S(t-s)B\Big[W^{-1}\int_0^m S(m-\tau)[f(\tau) - \overline{f}(\tau)]d\tau\Big](s)ds \Big| \\
&+\frac{1}{2} \Big| \int_0^t S(t-s)B\Big[\overline{W}^{-1}\int_0^m C(m-\tau)[f(\tau) - \overline{f}(\tau)]d\tau\Big](s)ds \Big| \\
&\leq \widetilde{M}m\Big[\|x_t - \overline{x}_t\| + |x'(t) - \overline{x'}(t)|\Big]\int_0^m [l(s) + l'(s)]ds \\
&+\frac{1}{2}\int_0^t |S(t-s)|M_3M_1\widetilde{M}m\Big[\|x_t - \overline{x}_t\| + |x'(t) - \overline{x'}(t)|\Big]\int_0^m [l(\tau) + l'(\tau)]d\tau \\
&+\frac{1}{2}\int_0^t |S(t-s)|M_3M_2Mm\Big[\|x_t - \overline{x}_t\| + |x'(t) - \overline{x'}(t)|\Big]\int_0^m [l(\tau) + l'(\tau)]d\tau \\
&\leq m\widetilde{M}L\Big[\|x_t - \overline{x}_t\| + |x'(t) - \overline{x'}(t)|\Big] \\
&+\frac{1}{2}\widetilde{M}M_3m(M_1\widetilde{M}m + M_2mM)L\Big[\|x_t - \overline{x}_t\| + |x'(t) - \overline{x'}(t)|\Big] \\
&\leq m\widetilde{M}L[1 + (1/2)M_3(M_1\widetilde{M}m + M_2mM)](\|x_t - \overline{x}_t\| + |x'(t) - \overline{x'}(t)|),
\end{aligned}
$$

where $L = \int_0^m [l(s) + l'(s)]ds$. Then

$$\|x(t) - \overline{x}(t)\| \leq m\widetilde{M}L[1 + (1/2)M_3m(M_1\widetilde{M} + M_2M)](\|x_t - \overline{x}_t\| + |x'(t) - \overline{x'}(t)|).$$

By interchanging the roles of x and $\overline{x}$, we get an analogous relation

$$H_d(G_1(y), G_2(\overline{y})) \leq m\widetilde{M}L[1+(1/2)M_3m(M_1\widetilde{M}+M_2M)](\|x_t-\overline{x}_t\|+|x'(t)-\overline{x'}(t)|).$$

Similarly, for any x' and $\overline{x'}$, we obtain

$$H_d(G_1(y'), G_2(\overline{y'})) \leq m\widetilde{M}L[1+(1/2)M_3m(M_1\widetilde{M}+M_2M)](\|x_t-\overline{x}_t\|+|x'(t)-\overline{x'}(t)|).$$

Thus G_1 and G_2 are contractive. As a consequence of Lemma 2.3, we deduce that G_1 and G_2 have a fixed point. Hence, system (1) is controllable on J.

4. Conclusion

In this paper, we discussed the controllability results for the second order impulsive neutral functional integrodifferential inclusions, with infinite delay through phase space defined in Section 2. We have proved the result without compactness of family of cosine operators. The result is obtained using a Hausdorff space (defined in Section 2) and a contraction mapping for multi-valued maps due to [11].

References

[1] Balachandran, K. Leelamani, A., Kim,J.H, *Controllability of Neutral Functional Evolution Integrodifferential Systems with Infinite Delay*, IMA J. Math. Control and Information, 25 (2) (2008), 157-171.

[2] Balachandran, K., Marshal Anthoni, S., *Controllability of Second Order Semilinear Ordinary Differential Systems in Banach Spaces*, J. Appl. Math. Stochastic Anal., 12 (2) (1999), 265-277.

[3] Balachandran, K., Park, D.G., Manimegalai, P., *Controllability of Second-Order Integrodifferential Evolution Systems in Banach Spaces*, Comput. and Math. Appl., 49 (2005), 1623-1642.

[4] Banas, J., Goebel, K., *Measures of Noncompactness in Banach Spaces,* Dekker, New York, 1980.

[5] Castaing, C., Valadier, M., *Convex Analysis and Measurable Multi functions*, Lecture Notes in Mathematics, vol. 580, Springer, Berlin, 1977.

[6] Chalishajar, D.N., Acharya, F.S., *Controllability of first order neutral impulsive differential inclusions with nonlocal conditions*, App. Math., 2 (2011), 1486-1496.

[7] Chalishajar, D.N., Acharya, F.S., *Controllability of second order semi-linear neutral impulsive differential inclusions on unbounded domain with infinite delay in Banach spaces*, Bull. Korean Math. Soc., 48 (2011), 813-838.

[8] Chalishajar, D., *Controllability of second order impulsive neutral functional differential inclusions with infinite delay*, J Optim. Theory Appl. 154 (2012), 672-684.

[9] Chang, Y.K., Li, W.T., *Controllability of functional integro-differential inclusion with an unbounded delay*, J. Optim. Theory Appl., 32 (2007), 125-142.

[10] Chang, Y.K., Anguraj,A., Mallika Arjunan, M., *Controllability of impulsive neutral functional differential inclusions with infinite delay in Banach spaces*, Chaos, Solitons and Fractals, 39 (2009), 1864-1876.

[11] Covitz, H., Nadler Jr., S.B., *Multivalued contraction mapping in generalised metric spaces*, Isr. J. Math., 8 (1970), 5-11.

[12] Deimling, K., *Multivalued Differential Equations. de Gruyter*, Berlin, 1992.

[13] Goldstein, J.A., *Semigroups of Linear Operators and Applications*, Oxford University Press, New York, 1985.

[14] Gunasekar, T. Paul Samuel, F., Mallika Arjunan, M., *Existence of solutions for impulsive Partial neutral functional evolution Integrodifferential inclusions With infinite delay*, Int. J. Pure and Appl. Math., 85 (2013), 939-954.

[15] Hale, J.K., Kato, J., *Phase space for retarded equations with infinite delay*, Funkc. Ekvacioj., 21 (1978), 11-41.

[16] Hernandez, E., Rabello, M., Henriquez, H.R., *Existence of solutions for impulsive partial neutral functional differential equations*, J. Math. Anal. Appl., 331 (2007), 1135-1158.

[17] Hino,Y., Murakami, S., Naiito, T., *In Functional-differential equations with infinite delay,* Lecture notes in Mathematics, 1473, Springer-Verlog, Berlin, 1991.

[18] Hu, S., Papageogiou, N.S., *Handbook of Multivalued Analysis*, Kluwer Academic, Dordrecht, 1997.

[19] Klamka J., *Constrained controllability of semilinear delayed systems*, Bulletin of the Polish Academy of Sciences, Technical Sciences, 49 (3) (2001), 505-515.

[20] Klamka J., *Constrained controllability of semilinear systems*, Nonlinear Analysis, 47 (2001), 2939-2949.

[21] Klamka J., *Constrained exact controllability of semilinear systems*, Systems and Control Letters, 4 (2) (2002), 139-147.

[22] Lakshmikantham, V., Bainov, D.D., Simeonov, P.S., *Theory of Impulsive Differential Equations*, World Scientific, Singapore, 1989.

[23] Liu, B., *Controllability of impulsive neutral functional differential inclusions with infinite delay*, Nonlinear Anal. 60 (2005), 1533-1552.

[24] Triggiani, R., *Addendum: A note on lack of exact controllability for mild solution in Banach spaces*, SIAM J. Control Optim. 15 (1977), 407-411. MR 55 8942; SIAM, J. Control Optim. 18 (1) (1980), 98-99.

[25] Travis, C.C., Webb, G. F., *Cosine families and abstarct nonlinear second order differential equations*, Acta. Math. Acad. Sci. Hungaricae, 32 (1978), 76-96.

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